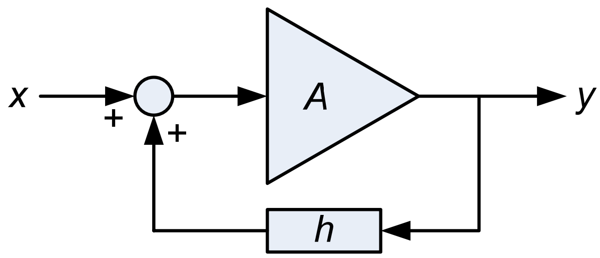
TechTest2007 (Selected) Solutions

5 March 2007(v7mar07)

6. Stability and Negative Feedback. Nature and Man use negative feedback to make systems ranging from sun-following flowers to electronic circuits work properly when some of their components (subsystems) degrade. In electronics we can take a portion of the output (y) of an amplifier, reverse its sign, and sum it with the external input (x) to create a more stable amplifier whose performance is less sensitive to the gain changes of the original amplifier. The essence of this system is shown in the diagram below.



Here the 'open-loop' gain of the amplifier is A. The feedback gain is h so that the quantity hy is summed with the x before being input to the amplifier. We see that when $h = 0$ then the open-loop output is just the expected $y = Ax$. We select $0 > h > -1$ to close the loop and create a negative feedback stabilized amplifier that now has a gain A_f with output $y = A_f x$.

a) Derive the formula for A_f as a function of A and h. (30%)
b) To obtain a feedback amplifier with $A_f = 50$, specify the nominal value of A given that $h = -0.01$. (10%)
c) Derive the (first-order) formula for computing the sensitivity of A_f to changes in A. (40%)
d) Using the sensitivity formula, demonstrate the gain in stability of the above feedback amplifier by computing the percent change in A_f when A degrades to 90% of its original value. (20%)

Solution

a) Input to amplifier A is $x - hy$, therefore the closed-loop output is

$$y = A(x - hy) \rightarrow \frac{y}{x} = A\left(1 - h\frac{y}{x}\right)$$

$$A_f = \frac{y}{x} = \frac{A}{1 - hA}$$

Note that $A_f(h{=}0) = A$ as expected.

b) Solving A_f for A gives $A = \frac{A_f}{1 + hA_f}$, and using $A_f = 50$ and $h = -0.01$ gives $A = 100$.

c) From the calculus, first order perturbations in a function $z = f(x)$ are computed from $\Delta z = \left.\frac{dz}{dx}\right|_x \Delta x$. In our case we seek ΔA_f as a function of ΔA, therefore taking the derivative and simplifying gives

$$\Delta A_f = \left.\frac{dA_f}{dA}\right|_A \Delta A = \left.\left[\frac{1}{1-hA} + \frac{hA}{(1-hA)^2}\right]\right|_A \Delta A = \left.\frac{1+hA_f}{1-hA}\right|_{A,A_f} \Delta A$$

d) Calculating $\Delta A = (0.1)100 = 10$ and substituting $A_f = 50$, $h = -0.01$, and $A = 100$ into the above gives

$$\Delta A_f = \left[\frac{1+(-0.01)50}{1-(-0.01)100}\right](-10) = \frac{0.5}{101}(-10) \approx -0.05$$

This yields the fractional change in the closed-loop gain of $\Delta A_f / A_f = -0.05/50 = -0.001$ or -0.1% as compared to $\Delta A / A = -0.1 = -10\%$ and demonstrates the stabilizing effect of negative feedback.

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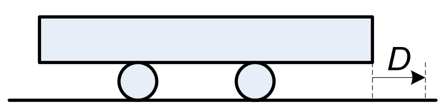
7. Moving A Slab on Rolling Logs. A flat stone slab is moved by placing it on parallel cylindrical logs of identical dimensions as shown in the figure below.



a) Derive the formula for calculating the distance D the slab moves relative to the ground as a function of the number n of revolutions its supporting logs of radius r roll on a flat surface. (80%)

b) How many feet forward will the slab move when supported by one foot diameter logs that roll 3.5 revolutions? (20%)

Solution

a) As the logs roll forward one full revolution, their centers move forward a distance $2\pi r$ that is equal to their circumference. At the same time the point of contact on the slab moves backward the same distance thereby pushing the slab forward two circumferences for every complete revolution of the logs. Therefore $D = 4n\pi r$.

b) $D = 4*3.5*\pi*0.5 = 21.99$ or almost 22 feet.

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8. Minimizing Perimeter to Area of a Rectangle. Derive the shape, expressed as the ratio *l/w*, of a rectangle of length *l* and width *w* that minimizes *P*/*A*, the ratio of its perimeter to area.

Solution

The approach here is to get student to express P/A as a function of the rectangle's single dimension by using the constraint as the formula for an arbitrary constant area (40%) and then calculate the required dimensions from setting the first derivative to zero and proving that the resulting rectangle is a square (40%). Proving that the zero derivative point is a minimum by showing that the second derivative is positive is worth the remaining 20%.

For an arbitrarily constant area *A* we proceed as follows:

$$A = lw \to l = \frac{A}{w} \to A = \left(\frac{A}{w}\right)w,$$

$$P = 2(l+w) = 2\left(\frac{A}{w} + w\right),$$

$$\text{Let } r = \frac{P}{A} = 2\left(\frac{1}{w} + \frac{w}{A}\right),$$

$$\frac{dr}{dw} = 2\left(\frac{-1}{w^2} + \frac{1}{A}\right) = 0 \to w^2 = A \to w = +\sqrt{A} \to l = \frac{A}{+\sqrt{A}} = +\sqrt{A} = w$$

$\therefore$ P/A minimizing rectangle is a square with l/w=1.

$$\frac{d^2r}{dw^2} = \frac{4}{w^3} > 0 \text{ for } w > 0, \quad \therefore \text{ above zero point is a minimum. QED}$$

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9. Gunfighters' Chances for Survival. Three probability impaired gunfighters have decided to settle their differences once and for all by having a three-way duel. Abe and Ben are sure shots who never miss their target, but Clem only manages to hit what he shoots at fifty percent of the time. To give Clem an 'equal chance' they all agree to fire one shot at each other sequentially in a predetermined random order until only one remains standing. Calculate the probability of survival for each of the gunfighters.

Solution. The only way for Abe or Ben to survive, if either of them goes first, is to shoot the other and leave Clem standing. Then if Clem misses with probability 0.5, the sure-shot lives since he next shoots Clem with probability 1. Therefore the survival probability for all three is 0.5 given that either Abe or Ben have the first turn to shoot. If Clem must shoot first, his optimal policy is to purposely miss (all gunfighters can hit the sky with probability 1), for if by chance he is to kill either Abe or Ben, then surely he will die next. By intentionally missing, Clem effectively resets the duel to having Abe or Ben as the starter and Clem knows he will survive that gunfight with probability 0.5. The case for the other two, however, is not so rosy since, in the randomly determined order, either one of them may be up next with the outcome as described above. This means that the complementary probability of 0.5 for Clem when he fires first, must be divided equally between Abe and Ben giving each of them a 0.25 chance of survival. (50% for correctly deriving the conditional probabilities) Finally, since the random firing order gives each the probability 1/3 of being first, for each the total probability of surviving is simply the sum of the conditional probabilities multiplied by the marginal probability of the condition occurring. The probability calculus is straightforward.

Let $P(X|Y)$ be the probability that X occurs given that Y is true. Then the joint probability $P(X,Y) = P(X|Y)P(Y)$. In our case X is the survival event AS, BS, or CS, and Y is the first shooter event AF, BF, or CF. Since the Y variables are mutually exclusive and independent, we can write the total survival probabilities by summing joint probabilities containing the survival event for each of the gunfighters.

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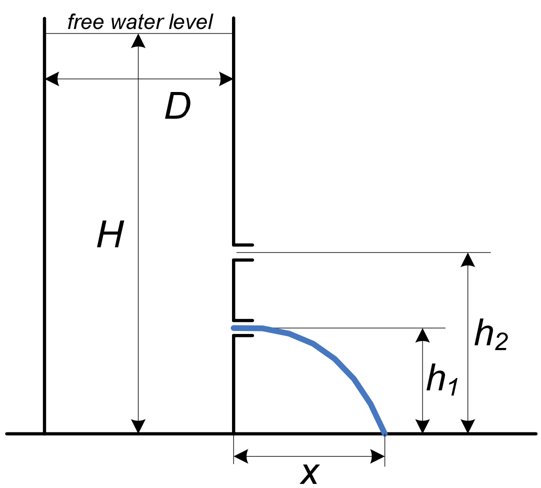
\begin{aligned}
P(AS) &= P(AS,AF) + P(AS,BF) + P(AS,CF) \\
&= P(AS|AF)P(AF) + P(AS|BF)P(BF) + P(AS|CF)P(CF) \\
&= 0.5 * 1/3 + 0 * 1/3 + 0.25 * 1/3 = 0.25 \\
P(BS) &= 0.25, \text{ same calculation as for } AS. \\
P(CS) &= P(CS,AF) + P(CS,BF) + P(CS,CF) \\
&= P(CS|AF)P(AF) + P(CS|BF)P(BF) + P(CS|CF)P(CF) \\
&= 0.5 * 1/3 + 0.5 * 1/3 + 0.5 * 1/3 = 0.5
\end{aligned}
$$

(50% for stating the correct total survival probabilities)



10. Two Holes in a Tank. Is it possible to locate and concurrently open two equal sized holes aligned vertically on the side of a full cylindrical tank filled with water to height H (as shown in figure) so that the issuing water streams will strike the ground at the same point? Prove your answer ignoring the effects of air pressure and resistance to motion.

Solution

The horizontal velocity v from a hole at height h is given by $k\sqrt{H-h}$ where k is a constant dependent on gravity. The horizontal distance covered by such a stream in time interval t is $vt = kt\sqrt{H-h}$. The vertical distance s the water (or anything) falls in the same

interval is calculated from $s = at^2/2$. If $s = h$ and $a = g$ (acceleration of gravity) then $h = gt^2/2$. Then from height h the water will hit the ground in time $t = \sqrt{\frac{2h}{g}}$ at a distance $x = vt = k\sqrt{H-h}\sqrt{\frac{2h}{g}} = k'\sqrt{Hh-h^2}$ from the tank. Since the expression under the radical is a quadratic there is the possibility of two values of h for $x>0$. (40% to this point)
Noting that x^2 and x are both maximum at the same value of h, we calculate that value from

$$\frac{d(x^2)}{dh} = k'^2(H-2h) = 0 \rightarrow h = \frac{H}{2}.$$

Relying on the symmetry of the quadratic about its maximum point, we examine holes placed at $h = H/2 \pm y$ for $y < H/2$. We then ask

$$x^2 = k'^2\left[H\left(\frac{H}{2}+y\right)-\left(\frac{H}{2}+y\right)^2\right] \stackrel{?}{=} k'^2\left[H\left(\frac{H}{2}-y\right)-\left(\frac{H}{2}-y\right)^2\right]$$

or $$\frac{H^2}{2}+Hy-\left(\frac{H}{2}\right)^2-Hy-y^2 \stackrel{?}{=} \frac{H^2}{2}-Hy-\left(\frac{H}{2}\right)^2+Hy-y^2$$

and discover that indeed $\frac{H^2}{2}+y^2 \equiv \frac{H^2}{2}+y^2$.

This proves that, as anticipated, two holes located at $h = H/2 \pm y$ will produce water streams that both hit the ground at exactly $x = k'\sqrt{\frac{H^2}{4}+y^2}$. (40%)

(For an additional 20% the student may conclude any of the following) The units of *k'* must be a pure numeric since the units in the radical are length squared. (This is true because $k' = \sqrt{2g}\sqrt{2/g} = 2$.) From this we conclude that under these conditions all liquids in any gravitational field will always fall the same distance $x(H,h)$ from the tank. Solving $x(H,h)$ for $H = f(x,h)$ and measuring x and h can now be used to determine the fluid level of a tank from

$$H = \left[\left(\frac{x}{k'}\right)^2+h^2\right]\frac{1}{h} = \frac{x^2}{k'h}+h = \frac{x^2}{2h}+h.$$