TechTest2008 Solution Key

Problem 1. [8 points] Solve 2sin$x$ – csc$x$ = 1 for all x(radians).

Solution. Multiply through by sin$x$ and rearrange to get $2\sin^2 x - \sin x - 1 = 0$. Factoring this gives $(2\sin x + 1)(\sin x - 1) = 0$ which results in the two conditions for the equation to hold. [50%]

From $2\sin x + 1 = 0$ we have sin$x$ = -0.5, this yields $x = 7\pi/6, 11\pi/6$. [25%]
From $\sin x - 1 = 0$ we have sin$x$ = 1, this yields $x = \pi/2$. [25%]

Then the solutions are $x = \pi/2, 7\pi/6, 11\pi/6$.

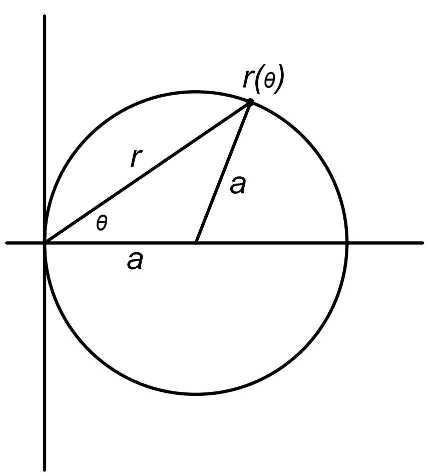
Problem 2. [10 points] Solve $2\cos^2(x/2) = \cos^2 x$ for all x (degrees).

Solution. Substitute $2\cos^2(x/2) = 1 + \cos x$ giving $\cos^2 x - \cos x - 1 = 0$ [25%] for which $\cos x = \left(\frac{1 \pm \sqrt{5}}{2}\right)$ = 1.6180, -0.6180. Since |cos| cannot exceed 1 [25%], we have cos$x$ = -0.6180 which yields the solutions $x = 128^o10'$, $231^o50'$ [50%]. Alternatively, take square roots of both sides which yields the two equations $\sqrt{2}\cos\frac{1}{2}x = \pm\cos x$. The 'plus solution' again is $231^o50'$, and the 'minus solution' is $128^o10'$.

Problem 3. [10 points] Solve the system of two equations for all $r > 0$ and $0 \le \theta < 2\pi$.

$$r\sin\theta = 2, \quad r\cos\theta = 3$$

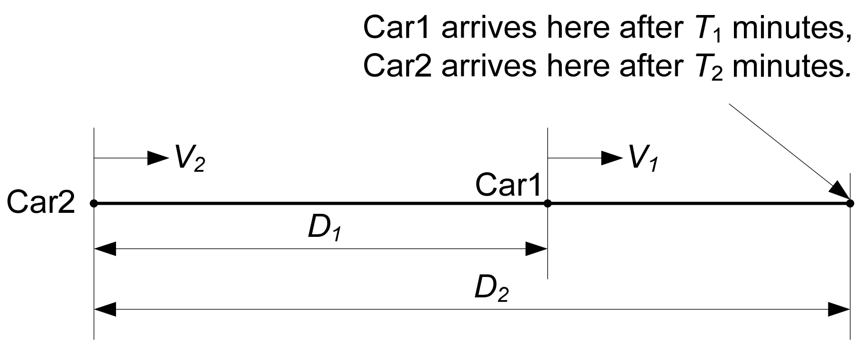
Solution. Squaring the equations and adding gives $r^2\sin^2\theta + r^2\cos^2\theta = r^2 = 13$ and r = 3.606. [25%] Now when $r > 0$, sinθ and cosθ must both be >0 and therefore θ is acute. [50%] Then dividing the first equation by the second gives $\tan\theta = 2/3 = 0.6667$ and θ = 0.5880 rads = $33^o41'$.[25%]



Problem 4. [10 points] Write the polar equation $r(\theta)$ of a circle with radius a and its center at $(a, 0)$.

Solution. Drawing the correct figure. [40%] Recalling and applying the law of cosines for any triangle: $a^2 = a^2 + r^2 - 2ar\cos\theta$ [30%] and solving $r^2 = 2ar\cos\theta$ for r yields the desired $r(\theta) = 2a\cos\theta$. [30%]

Problem 5. [15 points] The optimal constant driving speed V_2 is required for Car2 that must travel a distance D_2 in time T_2 while not passing Car1 that is a distance D_1 ahead of it and travelling at a constant speed $V_1 < V_2$. Both cars are travelling on the same straight line as shown in the figure and Car2 may not pass Car1. Find the optimal value V_2^* of V_2 such that the utility function $U(V_2) = wT_2 + (1-w)aV_2^2$ is minimized where $0 < w < 1$ and $a > 0$ are given constants. From physics we recognize that a will be dependent on the drag coefficient and the fuel consumption rate of the car, so minimizing the utility function seeks to trade off the attributes of speed against energy consumption – a common realworld optimization problem. Hint: Express U in terms of only V_2, then add a little calculus while keeping the physical situation firmly in mind. It's easier than it looks.



Solution. From the figure one can immediately write

$$\frac{D_2}{V_2} = T_2 \geq \frac{D_2 - D_1}{V_1} = T_1 = T_{2,\min} \Rightarrow V_2 \leq \left(\frac{D_2}{D_2 - D_1}\right)V_1 \qquad [10\%]$$

Substituting in U for T_2 gives utility as a function of only the desired variable V_2,

$$U(V_2) = w\frac{D_2}{V_2} + (1-w)aV_2^2 \qquad [15\%]$$

Since this is the function to be minimized, we proceed in the standard manner of setting the first derivative to zero, solving for V_2, and checking that this value yields a positive second derivative.

$$\frac{dU}{dV_2} = -w\frac{D_2}{V_2^2} + (1-w)2aV_2 = 0$$

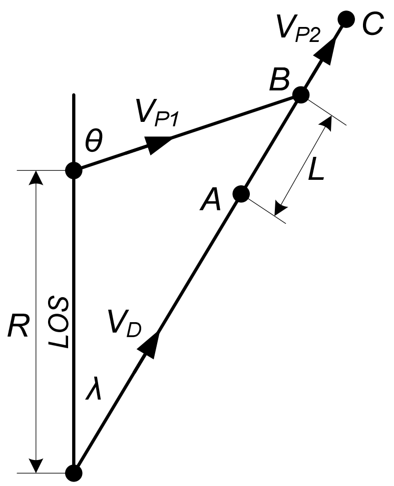
$$\frac{wD_2}{2a(1-w)} = V_2^3 \rightarrow V_2 = \left[\frac{wD_2}{2a(1-w)}\right]^{\frac{1}{3}}$$

Given the values of the constants, we are guaranteed that V_2 is both real and positive. [15%] To confirm that $U(V_2)$ is also a minimum we compute the inequality constraint on V_2 from the second derivative.

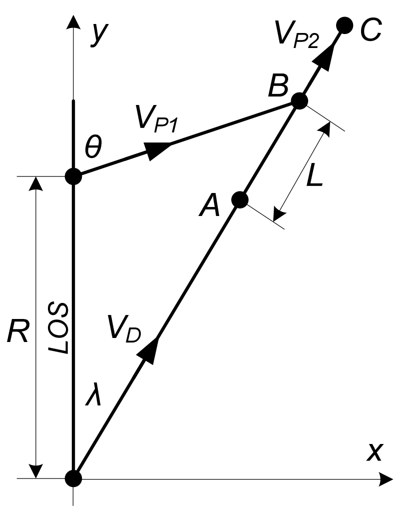
$$\frac{\partial^2 U}{\partial V_2{}^2} = -\frac{2wD_2}{V_2^3} + 2a(1-w) > 0 \rightarrow V_2^3 < \frac{wD_2}{a(1-w)}$$

From the above solution for V_2 we see by inspection that this constraint for a minimum is indeed satisfied. [20%] To determine whether our unconstrained minimizing solution is feasible, we have to remember that Car2 is not allowed to pass Car1 but both cars can arrive concurrently at the 'D_2 point'. This was expressed in the first constraint on V_2. Hence the optimal solution V_2^* is the minimum of the unconstrained solution and the 'no passing' speed constraint.

$$V_2^* = \min\left\{\left[\frac{wD_2}{(1-w)\,2a}\right]^{\frac{1}{3}}, \left(\frac{D_2}{D_2 - D_1}\right)V_1\right\} \qquad [40\%]$$



Problem 6. [15 points] A dolphin detects an unwary prey fish at a distance of R feet and swimming at speed V_{P1} in a straight line that makes and angle θ with the line of sound/sight (LOS) as shown in the figure. The dolphin knows that such a prey will become aware of being pursued when it comes within L feet of the prey (at point A), at which instant the prey will turn tail (at point B swimming directly away from the dolphin) and attempt to escape at $V_{P2} > V_{P1}$. With this knowledge the dolphin is able to instantly compute a proper lead angle λ and take off after its prey at a high constant pursuit speed $V_D > V_{P2}$. The proper lead angle is such that, given the above tactic, the dolphin will catch its prey (at point C) in the minimum time T. Assuming all accelerations are instantaneous and occur in 'the plane of action', derive the relations for λ and T in terms of the given problem parameters. Calculate λ and T for R = 50 ft, θ = 60 deg, L = 15 ft, V_{P1} = 3 ft/sec, V_{P2} = 15 ft/sec, V_D = 30 ft/sec. Remember the dolphin wants to eat as soon as possible. Hint: Use the Law of Cosines.



Solution. Let T_L be the time it takes for the dolphin to get to point A and the prey to get to point B. Then the sides of the triangle can be expressed in terms of T_L and used in the Law of Cosines to yield the relation to be solved for T_L. The minimum positive real value of T_L from the quadratic is the desired solution. Then we set up a convenient coordinate system [10%] and express the coordinates of point B in terms of λ which allows us to write the arctan relationship that yields λ. The final value of T is computed by adding to T_L the time it takes the dolphin to overtake the prey (when they each are at A and B respectively). The appropriate relations are below.

From the Law of Cosines

$$\begin{aligned}(V_D T_L + L)^2 &= R^2 + V_{P1}^2 T_L^2 - 2RV_{P1}T_L\cos(\pi-\theta)\\ &= R^2 + V_{P1}^2 T_L^2 + 2RV_{P1}T_L\cos\theta\end{aligned}$$ [15%]

From the solution to the quadratic in T_L

$$\left(V_D^2 - V_{P1}^2\right)T_L^2 + 2\left(LV_D - RV_{P1}\cos\theta\right) + \left(L^2 - R^2\right) = 0,$$ [5%]

which gives

$$T_L = \frac{\left(RV_{P1}\cos\theta - LV_D\right) \pm \left[\left(LV_D - RV_{P1}\cos\theta\right)^2 + \left(V_D^2 - V_{P1}^2\right)\left(R^2 - L^2\right)\right]^{1/2}}{\left(V_D^2 - V_{P1}^2\right)}.$$ [5%]

The coordinates of point B in terms of λ are

$$\begin{aligned}x_B &= V_{P1}T_L\sin\theta = \left(V_D T_L + L\right)\sin\lambda\\ y_B &= V_{P1}T_L\cos\theta + R = \left(V_D T_L + L\right)\cos\lambda\end{aligned},$$ [15%]

therefore

$$\tan\lambda = \frac{x_B}{y_B} = \left[\frac{V_{P1}T_L\sin\theta}{V_{P1}T_L\cos\theta + R}\right] \rightarrow \lambda = \tan^{-1}\left[\frac{V_{P1}T_L\sin\theta}{V_{P1}T_L\cos\theta + R}\right],$$ [10%]

which completes the solution for T_L and λ. The complete general solution requires the total time to intercept (lunch time) calculated from

$$T = T_L + \frac{L}{V_D - V_{P2}}$$ [20%]

Now substituting in the given numerical values yields

T_L = 1.23 sec, x_B = 3.20 ft, y_B = 51.85 ft, λ = 3.53 deg, T = 2.23 sec. [20%]

Problem 7. [10 points] The infinite series expansion of any positive number $a > 0$ raised to a finite power $|x| < \infty$ is

$$a^x = 1 + \frac{x\ln a}{1!} + \frac{(x\ln a)^2}{2!} + \ldots = \sum_{k=0}^{\infty}\frac{(x\ln a)^k}{k!},$$

and a very important relation defining operations in the complex plane is

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where $i^2 = -1$, $n! = n(n-1)(n-2)\ldots 1$ is the factorial where $0! = 1$, and ln is the natural logarithm. (BTW, the last relationship is verbalized and remembered as '*e* to the *i*-theta equals CIS theta'.) Given these relationships derive the infinite series expressions for sin θ and cos θ that are often used for fast small angle approximations for trigonometric functions.

Solution. Let $a = e$ and $x = i\theta$ [10%], then using the first relationship we write the series expansion

$$\begin{aligned} e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} \\ &= 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \ldots \\ &= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots \\ &= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\right] + i\left[\frac{\theta}{1!} - \frac{\theta^3}{3!} + \ldots\right] \\ &= \cos\theta + i\sin\theta \end{aligned} \qquad [40\%]$$

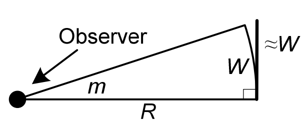
Therefore

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots, \quad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \ldots \ , \qquad [40\%]$$

or more formally

$$\cos\theta = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!}, \quad \sin\theta = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \theta^{2k+1}}{(2k-1)!}. \qquad [10\%]$$

Problem 8. [7 points] The Stadia Method enables the remote measurement of size or distance to objects by using natural angles (i.e. radians). The basic relationship is shown in the figure. There we see that $W = mR$ where W and R have units of length and angle m is measured in radians. For small m it is clear that the arc length of the subtended sector and the vertical side of the right triangle are almost equal.



a) Suppose you have calibrated your hand, and by extending your arm and pointing to some pine trees on a distant ridge you measure them to subtend an angle of 20 milliradians. The distant trees are about the same type and height that you are standing under which you estimate to be about 60 feet tall. What is your estimate of the distance to the ridge?
b) If you know your finger widths to within ±0.003 radians and the tree height to ±5 feet, knowing that your expected error of the distance to be quite a bit smaller, calculate the extreme error bounds on your estimate from part a).

Solution. a) R = 60 ft/0.020 radians = 3,000 feet. [30%] b) From the stadia relationship (also known as the 'Worm Formula' in field artillery) we calculate the max and min values for R as follows.

$$R_{\max} = \frac{(60+5)\,\text{ft}}{(0.020-0.003)} = 3{,}824 \text{ ft}$$

$$R_{\min} = \frac{(60-5)\,\text{ft}}{(0.020+0.003)} = 2{,}391 \text{ ft}$$

[70%]

Problem 9. [15 points] The Time Value of Money. An amount PV (Present Value) deposited for a year in a bank that pays an annual interest rate i will yield a future value FV computed by adding together the initial principal PV and the interest amount iPV, or $FV = PV + iPV = PV(1+i)$.

a) If the amount PV_0 is left on deposit for three years during which the bank pays varying annual interest rates of i_1, i_2, i_3 then show that after this period the account is worth $FV_3 = PV_0(1+i_1)(1+i_2)(1+i_3) = PV_0 \prod_{k=1}^{3}(1+i_k)$.

b) If the interest rate is kept constant at i for N years, using the result from a), write the compact expression for calculating FV_N which is known as the 'time value of money formula'.

c) Using the result from part b), if $3,000 is deposited for five years at which time the account amounted to $4,000, what constant annual rate of interest i did the bank pay over this interval?

d) Again using the result of b), for how many years must money be kept in an account that pays 5% annual interest for it to triple?

e) Can the average interest rate $\bar{i}$ paid over N years be used in the time value of money formula to accurately compute FV_N instead of using the continued product formula given in part a)? Prove your answer. Hint: Use

$$FV_N = PV_0 \prod_{k=1}^{N}(1+i_k), \quad \bar{i} = \frac{1}{N}\sum_{k=1}^{N} i_k$$

Solution. a) Write the three annual appreciations, substituting them in sequence.

$FV_1 = PV_0(1+i_1)$ which becomes PV_1,

$FV_2 = PV_1(1+i_2)$ which becomes PV_2,

$FV_3 = PV_2(1+i_3)$ that upon substitution yields the desired result.

[10%]

b) $FV_N = PV(1+i)^N$ [10%]

c) Solving the solution for part b) for i gives

$$i = \left(\frac{FV_N}{PV}\right)^{\frac{1}{N}} - 1 \xrightarrow{substituting} i = \left(\frac{4000}{3000}\right)^{\frac{1}{5}} - 1 = 0.0592 = 5.92\%$$ [20%]

d) Solving the part b) solution for N and substituting the numerical data yields

$$\frac{FV_N}{PV} = (1+i)^N \rightarrow \ln\left(\frac{FV_N}{PV}\right) = \ln\left[(1+i)^N\right] = N\ln(1+i)$$
$$\therefore N = \frac{\ln\left(\frac{FV_N}{PV}\right)}{\ln(1+i)} \rightarrow \frac{FV_N}{PV} = 3 \rightarrow N = \frac{\ln(3)}{\ln(1+0.05)} = 22.52 \text{ years}$$ [30%]

e) No, since it can be shown with $i_1 \neq i_2$ that $(1+i_1)(1+i_2) \neq (1+\bar{i})^2 = \left(1+\frac{i_1+i_2}{2}\right)^2$ by expanding both sides of this inequality as follows.

$$1 + i_1 + i_2 + i_1 i_2 \overset{?}{=} 1 + (i_1 + i_2) + \frac{(i_1+i_2)^2}{4}$$
$$\overset{?}{=} 1 + i_1 + i_2 + \frac{i_1^2 + 2i_1 i_2 + i_2^2}{4},$$
$$i_1 i_2 \overset{?}{=} \frac{i_1^2 + 2i_1 i_2 + i_2^2}{4}$$
$$0 \overset{?}{=} \frac{i_1^2 - 2i_1 i_2 + i_2^2}{4},$$

Since $i_1^2 - 2i_1 i_2 + i_2^2 \neq 0$ for all i_1 and i_2, therefore the answer is NO. [30%]