TechTest2009 Solution Key

Problem 1. [9 points] a) What is the product of the first five prime numbers? b) Why is the product of the first two or more prime numbers always even?

Solution. a) The product of first five prime numbers is 2*3*5*7*11 = 2310 (recall that one is not a prime number). [40%] b) Because the first prime is two and the product of any natural number multiplied by two (or any even number) is even. [60%]

Problem 2. [9 points] Prove the following trigonometric identities.

a) $\sec^2\theta\csc^2\theta = \sec^2\theta + \csc^2\theta$

b) $\csc x = \frac{1}{2}\left[\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x}\right]$

Solution. a)

$$\sec^2\theta + \csc^2\theta = \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta\cos^2\theta} = \csc^2\theta\sec^2\theta$$ [50%]

b)

$$\left[\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x}\right] = \frac{\sin^2 x + (1+\cos x)^2}{\sin x(1+\cos x)} = \frac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{\sin x(1+\cos x)}$$
$$= \frac{2+2\cos x}{\sin x(1+\cos x)} = \frac{2(1+\cos x)}{\sin x(1+\cos x)} = 2\csc x, \quad \text{QED}$$ [50%]

Problem 3. [15 points] In error analysis the uncertainty δf in the output of a multi-variate model $f(\underline{x})$ is given by

$$\delta f(\underline{x})\Big|_{\underline{x}_0} = \left[\sum_{i=1}^{N}\left(\frac{\partial f}{\partial x_i}\delta x_i\right)^2\Bigg|_{\underline{x}_0}\right]^{\frac{1}{2}}, \quad \underline{x} = \left[x_1, x_2, \ldots, x_N\right],$$

where $\underline{x}$ is the vector notation of the set of independent variables in the model and the δx_i are their uncertainties. $\underline{x}_0$ denotes the 'point' in the variable space at which the model function f and its partial derivatives are evaluated. So the above formula for uncertainty holds at the point $\underline{x}_0$.

Suppose you are to determine the acceleration of gravity g (cm/s^2) using the well-known pendulum experiment which gives the period T (s) of the pendulum of length l (cm) as $T = 2\pi\sqrt{\frac{l}{g}}$ (the model). In the experiment you measure $l = 90.52 \pm 0.1$ cm, and $T = 1.910 \pm 0.005$ sec. Express your answer as $g \pm \delta g$ and evaluate the result in light of the usually accepted value of $g = 981$ cm/s^2.

Solution. The first task is to express the given model in the necessary form $g = g(l,T)$, and then write the expression for δg from the above discussion.

$$g = \frac{4\pi^2 l}{T^2} \rightarrow \frac{\partial g}{\partial l} = \frac{4\pi^2}{T^2}, \quad \frac{\partial g}{\partial T} = -\frac{8\pi^2 l}{T^3}$$ [35%]

Using the problem notation, we are given $\underline{x}_0 = [l, T] = [90.52, 1.910]$, $\delta l = 0.1$ cm, and $\delta T = 0.005$ sec. The expression for δg and its evaluation can now be written directly.

$$g(\underline{x}_0) = \left.\frac{4\pi^2 l}{T^2}\right|_{\underline{x}_0} = \frac{4\pi^2 90.52}{1.910^2} = 979.6\text{cm/s}^2$$

$$\left.\frac{\partial g}{\partial l}\right|_{\underline{x}_0} = \left.\frac{4\pi^2}{T^2}\right|_{\underline{x}_0} = \frac{4\pi^2}{1.910^2} = 10.82\ \text{s}^{-2}, \quad \left.\frac{\partial g}{\partial T}\right|_{\underline{x}_0} = -\left.\frac{8\pi^2 l}{T^3}\right|_{\underline{x}_0} = -\frac{8\pi^2 90.52}{1.910^3} = 1026\ \text{cm/s}^3$$ [45%]

Substituting this into the given uncertainty formula gives the desired answer

$$\left.\delta g\right|_{\underline{x}_0} = \left.\left[\left(\frac{\partial g}{\partial l}\delta l\right)^2 + \left(\frac{\partial g}{\partial T}\delta T\right)^2\right]^{\frac{1}{2}}\right|_{\underline{x}_0} = \left[(10.82 * 0.1)^2 + (1026 * 0.005)^2\right]^{\frac{1}{2}} = 5.243\ \text{cm/s}^2$$

$$\therefore g = 979.6 \pm 5.2\ \text{cm/s}^2$$

In light of the 'usual' value of $g = 981$ cm/s^2 and given the calculated uncertainty bracket, the result of the experiment is accepted. [20%]

Problem 4. [10 points] The total tax burden is desired for a taxpayer who pays income taxes and buys something that has a sales tax. Let E be the dollars earned on which an income tax rate of T_I is levied. Let P be the price of something subject to a sales tax rate of T_S. Under this system of taxation how much does the taxpayer have to earn in order to complete the desired purchase – i.e. what is the total tax burden? Also, derive the formula for the incremental increase in earnings ΔE required for an incremental increase in the income tax rate ΔT_I. Use your results to answer both questions when P = \$100, T_I = 20%, T_S = 10%, and ΔT_I = 1%. (Hint: see problem #3)

Solution. The after-tax spendable earnings are $E(1- T_I)$, and the total expense for a purchased item is $P(1+ T_S)$. To purchase the item, the spendable earnings must equal its total expense. Therefore

$$E(1-T_I) = P(1+T_S)$$

$$E = P \underbrace{\left(\frac{1+T_S}{1-T_I}\right)}_{\text{Tax Burden Factor}} = \$100\left(\frac{1+0.1}{1-0.2}\right) = \$100(1.3750) = \$137.50$$ [25%]

as the amount that must be earned to effect the purchase.. The incremental earnings increase ΔE to support an incremental income tax increase ΔT_I to still be able to purchase the item is computed from basic calculus as

$$\Delta E = \frac{dE}{dT_I}\Delta T_I = \left[\frac{P(1+T_S)}{(1-T_I)^2}\right]\Delta T_I = \left[\frac{E}{(1-T_I)}\right]\Delta T_I,$$ [35%]

where the derivative is evaluated at the current parameter values.

From the first equation the tax burden factor is 1.3750, or 37.5% is the total burden for acquiring the item. [20%] From the second equation, in order to purchase the same item, the added earnings increment to support the tax increase is

$$\Delta E = \left[\frac{\$137.50}{1-0.2}\right]0.01 = \$1.72,$$ [20%]

or earnings must increase by 1.25% to support a income tax increase of 1%.

Problem 5. [8 points] Conversion of quantities from one set of units to another is easy if you remember that conversion is nothing more than multiplying by a dimensionless unity expressed properly and treating all unit labels as algebraic variables. To convert centimeters to inches, we recall that 2.54 cm = 1 inch which allows to express unity in two different ways –

$$\frac{2.54(\text{cm})}{1(\text{inch})} = \frac{1(\text{inch})}{2.54(\text{cm})} = 1\ (\text{pure numeric}).$$

Therefore, to convert 7 cm into inches, we write

$$x(\text{inch}) = 7(\text{cm})*1 = 7(\text{cm})*\left[\frac{1}{2.54}*\frac{\text{inch}}{\text{cm}}\right] = \frac{7}{2.54}*\text{inch=2.76 inches}$$

a) What is the acceleration of gravity g in furlongs per fortnight squared? One furlong is one eighth of a mile, and a fortnight is two weeks. Take 1g = 32.174 ft/s^2.

b) In automotive engineering the conversion of torque *T,* delivered by an engine rotating at a given *RPM,* to horsepower *HP* is a standard mechanics problem in physics. Derive the formula that computes horsepower in terms of torque expressed in ft*lbs and engine rotation in revolutions per minute. Recall that 1 horsepower = 550 ft*lbs/sec, and that rotational power is $T\omega$ where ω is the angular velocity in radians/sec.

Solution. a) One mile = 5280 ft = 8 furlongs, therefore 1 furlong = 660 ft. Two weeks = 14 days, 1 day = 24 h, 1 h = 60 min, 1 min = 60 sec. Therefore

$$1g=\frac{32.174*\text{ft}}{\sec^2}\left\{\frac{\dfrac{1*\text{furlong}}{660*\text{ft}}}{\left[\left[\dfrac{1*\text{fortnight}}{2*\text{week}}\right]\left[\dfrac{2*\text{week}}{14*\text{day}}\right]\left[\dfrac{1*\text{day}}{24*\text{hr}}\right]\left[\dfrac{1*\text{hr}}{60*\text{min}}\right]\left[\dfrac{1*\text{min}}{60*\text{sec}}\right]\right]^2}\right\}$$

$$=\frac{32.174*\text{ft}}{\sec^2}\left\{\frac{\dfrac{1*\text{furlong}}{660*\text{ft}}}{\left[\dfrac{1*\text{fortnight}}{1,209,600*\sec}\right]^2}\right\}=\left\{\frac{32.174*1,209,600^2*\text{furlong}}{660*\text{fortnight}^2}\right\}$$

$$=\frac{7.1325*10^{10}*\text{furlong}}{\text{fortnight}^2}$$

[35%]

b) Using 2π radians = 1 revolution and that the radian is a pure numeric, we have

$$HP=T(\text{ft*lbs})*RPM*\left(\frac{\text{rev}}{\text{min}}\right)*\left(\frac{\text{min}}{60*\sec}\right)\left(\frac{2\pi*\text{radians}}{1*\text{rev}}\right)*\left(\frac{1*\text{HP}}{550\,\dfrac{\text{ft*lbs}}{\sec}}\right)$$

$$=T*RPM*\left(\frac{1}{60}\right)*2*\pi*\left(\frac{1*\text{HP}}{550}\right)=T*RPM*\left(\frac{2*3.14159*\text{HP}}{60*550}\right)$$

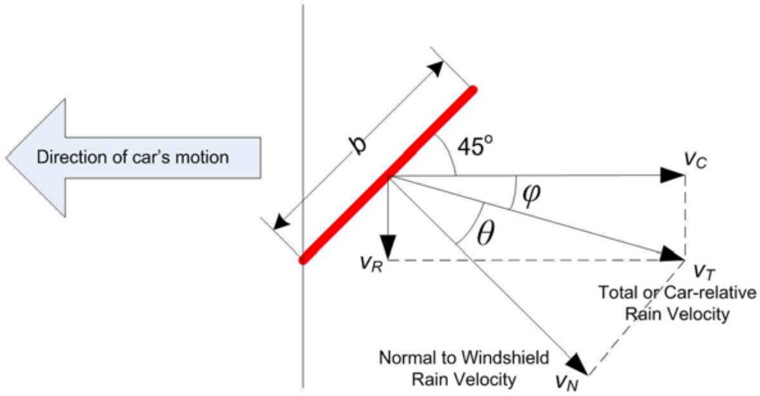
$$=\frac{T*RPM}{5252}(\text{HP})$$

[65%]

Where *HP*, *T*, and *RPM* are the numerical values of the respective terms.

Problem 6. [15 points] Rain through a missing windshield. a) What amount of rain (liters/sec) comes through a rectangular hole, (c meters wide, b meters high) that is tilted at 45 deg from vertical, when the car moves horizontally at a speed of v_C (m/s) as the rain is falling straight down at v_R (m/s), and has a uniform density of ρ (l/m^3)? b) Suppose c = 1.7 m, b = 0.5 m, v_R = 30 m/s, and ρ = 0.05 l/m^3. If the car can reach a maximum speed of $v_{C,max}$ = 200 km/h, at what speed v_C^* m/s (using the star to indicate an extremizing value) in that range should it be driven so as to maximize the amount of rain $\dot{l}_R^*$ l/s (using the dot symbol to indicate time rate of change) coming through the windshield hole? What is the value of $\dot{l}_R^*$ l/s?

Solution. a) The geometry of the problem is given in the figure below.



Drawing a useful figure [30%].

From the figure we then have the following vertical plane derivation.

$$45+\varphi+\theta=90, \quad \theta=45-\varphi$$

$$v_T=\sqrt{v_C^2+v_R^2}, \quad \varphi=\tan^{-1}\frac{v_R}{v_C}$$

$$v_N=v_T\cos\theta \qquad \text{[30\%]}$$

$$\text{Area of hole } A=bc,$$

$$\text{Rate of water through hole } \frac{dl_W}{dt}=\dot{l}_W=\rho A v_N=\rho A\sqrt{v_C^2+v_R^2}\cos(45-\varphi)$$

b) From the last line it is clear that the water flow through the windshield hole increases as the car's speed v_C increases. Therefore $\dot{l}_R^*$ is achieved at $v_C^*=v_{C,\max}$. (And for $v_C >> v_R$ the flow rate $\dot{l}_R^*$ approaches the limit $\frac{\rho A v_C}{\sqrt{2}}$.) [10%] Finally, converting $v_{C,max}$ to m/s and using the given values for the parameters to compute $\dot{l}_R^*$ yields

$$\theta=45-\varphi$$

$$v_T=\sqrt{v_{C,\max}^2+v_R^2}=\sqrt{(200,000/(60*60))^2+30^2}=63.14 \text{ m/s},$$

$$\varphi=\tan^{-1}\frac{v_R}{v_C}=\tan^{-1}\frac{30.00}{55.56}=28.37 \text{ deg}$$

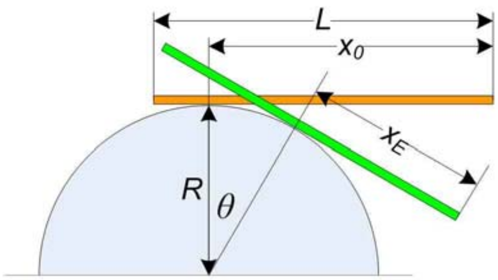
$$v_N=v_T\cos\theta=63.14*\cos(45-28.37)=63.14\cos(16.63)=60.50 \text{ m/s} \qquad \text{[30\%]}$$

$$\text{Area of hole } A=bc=1.7*0.5=0.85 \text{ m}^2,$$

$$\dot{l}_W=\rho A v_N=\rho A\sqrt{v_{C,\max}^2+v_R^2}\cos(45-\varphi)$$

$$=0.05*0.85*60.50*\cos(16.63)=2.46 \text{ liters/sec}$$

Problem 7. [12 points] Few people notice that it is easy to balance a stick on a ball, and fewer inquire why that is so. Just place the stick on the ball so that its approximate center point touches near the top of the ball, and the stick will automatically tilt in such a way that it comes to a point of equilibrium at an angle to the horizontal. Suppose you have a stick of constant linear density ρ (kg/m) that is L meters long. You also have a sphere (ball) of radius R (m) made of a non-deforming substance such that it prevents the stick from sliding along its surface. You place the stick on the top of the ball so that its original tangent point is x_0 from one end, and you slowly release the stick so that it comes to rest (equilibrium) at some angle θ from the vertical. Let $x_0 > L/2$ with no loss of generality. The figure shows the problem set up with the original and equilibrium positions of the stick in the 'plane of action'.



a) Derive the formula that expresses the angle θ and the length x_E to the tangent point at equilibrium in terms of the given parameters. b) Discuss how θ and x_E vary as x_0 approaches L. Remember that the stick cannot slide on the ball as long as $\theta < 90^\circ$. c) Compute the value of θ and x_E for $L = 1.5$ m, $R = 1.1$ m, $\rho = 3$ kg/m, $x_0 = 0.9$ m. Hint: Moments and torques are measured the same way.

Solution. a) The equilibrium point is reached when the moments at the tangent point balance each other. Within certain limits this system is stable in the sense that if the ball is rotated in the plane of action (in which the stick lays) then the negative feedback from the unbalanced moments is such that the stick seeks out a new value of θ and its corresponding x_E where the moments are again opposite and equal. The solution to this problem is simpler than it looks. We start by writing down the relationship between x_0 and x_E where θ is in radians.

$$x_0 = R\theta + x_E > L/2$$

$$x_E = x_0 - R\theta \rightarrow \theta = \frac{x_0 - x_E}{R}$$

Then balancing the gravity induced moments around the equilibrium tangent point, we have

$$\rho g\left(L - x_E\right)\left(\frac{L - x_E}{2}\cos\theta\right) = \rho g x_E\left(\frac{x_E}{2}\cos\theta\right)$$

$$x_E^2 = \left(L - x_E\right)^2 = L^2 - 2Lx_E + x_E^2 \rightarrow x_E = \frac{L}{2}$$

This demonstrates what should be obvious by inspection, that the equilibrium point is independent of θ and always occurs at the half-way point on the stick. [40%]

b) Substituting the solution for x_E into the above equation for θ gives

$$\theta = \frac{x_0 - L/2}{R} = \frac{2x_0 - L}{2R} < \frac{\pi}{2}$$
$$\lim_{x_0 \to L} \theta = \theta_{\max} = \frac{L}{2R} < \frac{\pi}{2} \Rightarrow L < \pi R,$$

which says that for sticks shorter than half the circumference of the ball, the stick will rotate to an equilibrium point whose maximum angle is less that $\pi/2$ or 90 degrees as the original tangent point x_0 approaches one end of the stick. If $L > \pi R$, then we can derive the maximum value of x_0, its related x_E, and θ from the constraint that the equilibrium doesn't exceed $\pi/2$ (and the stick drops off the ball).

$$\frac{2x_{0,\max} - L}{2R} \le \frac{\pi}{2} \to x_{0,\max} \le \frac{L + \pi R}{2}$$
$$\therefore x_{E,\max} = x_{0,\max} - R\frac{\pi}{2} = \frac{L+\pi R}{2} - \frac{\pi R}{2} = \frac{L}{2}, \text{ and confirming}$$
$$\theta_{\max} = \frac{1}{R}\left[\frac{L+\pi R}{2} - \frac{L}{2}\right] = \frac{\pi}{2}.$$

[50%]

c) From the above development we can immediately say that $x_E = 0.75$ m, and

$$\theta = \frac{x_0 - x_E}{R} = \frac{0.9 - 0.75}{1.1} = 0.136 \text{ rad.}$$

[10%]

Problem 8. [12 points] In an experiment, a biomass is known to go through three phases of growth. The fractional weight changes during each phase are given by f_1, f_2, f_3 such that in phase i the mass increases by $\Delta m_i = f_i m_{i-1}$ where m_{i-1} was the mass at the end of phase i-1. a) What is the total fractional weight gain F at the end of the experiment, i.e. at the end of phase three? b) Suppose F was found to be in error by an amount ΔF, and this error was isolated to a measurement error Δf_2 in f_2. Express Δf_2 in terms of the given parameters and simplify your answer. c) In the experiment you measured $f_1 = 0.07$, $f_2 = 0.13$, $f_3 = -0.05$, and $\Delta F = -0.02$, what is Δf_2?

Solution. a) The biomass at the end of phase i is given by $m_i = (1+f_i)\, m_{i-1}$. Therefore the total gain fraction F for the experiment is derived from

$$1 + F = (1+f_1)(1+f_2)(1+f_3)$$
$$F = (1+f_1)(1+f_2)(1+f_3) - 1.$$

[30%]

b) Let the correct fractional gain be $F' = F + \Delta F$. Then using the previous result, we have

$$1+F' = 1+F+\Delta F = \left(1+f_1\right)\left(1+f_2+\Delta f_2\right)\left(1+f_3\right)$$

$$1+f_2+\Delta f_2 = \frac{1+F+\Delta F}{\left(1+f_1\right)\left(1+f_3\right)}$$ [40%]

$$\Delta f_2 = \frac{1+F+\Delta F}{\left(1+f_1\right)\left(1+f_3\right)} - \left(1+f_2\right).$$

Now multiplying the first r.h.s term by $(1+f_2)/\ (1+f_2)$ and substituting as appropriate yields the more elegant and useful form.

$$\Delta f_2 = \left[\frac{1+F'}{1+F}\right]\left(1+f_2\right) - \left(1+f_2\right) = \left[\frac{1+F'}{1+F} - 1\right]\left(1+f_2\right) = \left[\frac{\Delta F}{1+F}\right]\left(1+f_2\right), \text{ or}$$

$$\Delta f_2 = \left[\frac{1+f_2}{1+F}\right]\Delta F$$ [10%]

.

This says that the measurement error for a phase is simply proportional to the total growth error scaled by the pro-rata contribution factor from the phase in question.

c) Substituting the given numerical values into the above relations for F and Δf_2, we have

$$F = (1+0.07)(1+0.13)(1-0.05) - 1 = 0.15$$

$$\Delta f_2 = \left[\frac{1+0.13}{1+0.15}\right](-0.02) = -0.02$$ [20%]

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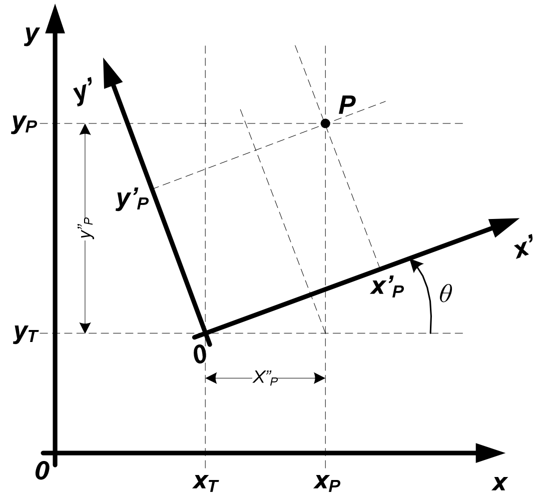
Problem 9. [10 points] A point (x_P, y_P) in the original (unprimed) cartesian coordinate system is mapped into a new (primed) cartesian coordinate system which is obtained by translation then rotation from the original system. a) If the origin of the primed system is translated by x_T and y_T in the unprimed system, and then rotated through the CCW angle θ, draw the appropriate figure and derive the relations for the coordinates in the primed system (x'_P, y'_P) in terms of the transformation variables and the original coordinates (x_P, y_P). b) For $(x_P, y_P) = (2, 1)$, $x_T = 5$, $y_T = 3$, $\theta = 10^{\circ}$, what are values for (x'_P, y'_P)?

Solution. a) An appropriate figure is shown here. [25%] First, the translated coordinates of P into the unrotated ($\theta = 0$) primed system are given by

$$x''_P = x_P - x_T, \quad y''_P = y_P - y_T.$$



Then the coordinate axes are rotated CCW through θ to yield the desired result in the translated then rotated system. [60%]

$$x'_P = x''_P \cos\theta + y''_P \sin\theta$$
$$y'_P = -x''_P \sin\theta + y''_P \cos\theta, \text{ or}$$
$$x'_P = (x_P - x_T)\cos\theta + (y_P - y_T)\sin\theta$$
$$y'_P = -(x_P - x_T)\sin\theta + (y_P - y_T)\cos\theta.$$

b) Substituting the given numerical values gives

$$x'_P = (2-5)\cos 10^\circ + (1-3)\sin 10^\circ = -3.3017$$
$$y'_P = -(2-5)\sin 10^\circ + (1-3)\cos 10^\circ = -1.4487$$

[15%]