TechTest2010

Merit Scholarship Examination
in the Sciences and Mathematics
given on 27 March 2010

sponsored by

The Sierra Environmental Studies Foundation

Solution Key

Problem 1. [5 points]

A) Solve tan $4x$ = cot $6x$ for all unique $x > 0$. [50%]
B) Solve the simultaneous system $r\sin\alpha = 2$, $r\cos\alpha = 3$ for all $r > 0$ and $0 \le \alpha < 2\pi$. [50%]

Solution: A) Recall that $\tan\theta$ has the period π, i.e. $\tan\theta = \tan(\theta + n\pi)$, $n = 0, \pm1, \pm2, \ldots$.
Since $\cot 6x = \tan(\pi/2 - 6x)$, we consider $\tan 4x = \tan(\pi/2 - 6x) = \tan(\pi/2 - 6x + n\pi)$. Equating the arguments yields $4x = (n + ½)\pi - 6x \rightarrow x = (n + ½)\pi/10 = \pi/20, 3\pi/20, 5\pi/20, \ldots, 39\pi/20$. [50%]

B) Square both sides of each equation and add them, giving $r^2(\sin^2\alpha + \cos^2\alpha) = 13 \rightarrow$ $r = +\sqrt{13} = 3.606$. When $r > 0$, $\sin\alpha$ and $\cos\alpha$ are both > 0 and α is acute. Dividing the first equation by the second gives $\tan\alpha = 2/3$ and $\alpha = 33^o41' = 0.5880$ radians. [50%]

Problem 2. [6 points]

Suppose a base amount A is successively increased by P_1 percent, which resulting amount is then increased by P_2 percent. What percent P_3 must this amount then be changed to achieve a net change of P_N from the base amount? [80%] Using your result, what value of P_3 would satisfy if $P_1 = 0.3$, $P_2 = 0.4$, and $P_N = -0.1$? [20%]

Solution: After the first two percent changes we have the amount $A_2 = A(1+ P_1)(1+ P_2)$. The desired net change amount is $A_N = A(1+ P_N)$ which can also be computed from A_2 by $A_N = A_2\,(1+ P_3)$. Substituting from these two relations gives

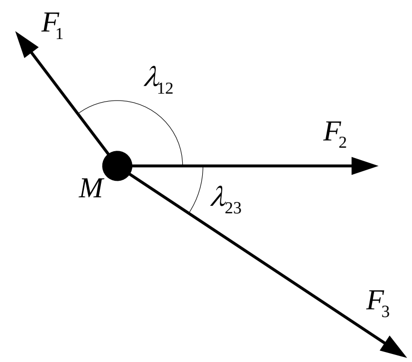
$$\begin{aligned} A_N &= A_2\left(1+P_3\right) \\ A\left(1+P_N\right) &= A\left(1+P_1\right)\left(1+P_2\right)\left(1+P_3\right) \\ \therefore\quad P_3 &= \frac{1+P_N}{\left(1+P_1\right)\left(1+P_2\right)} - 1 \end{aligned}$$ [80%]

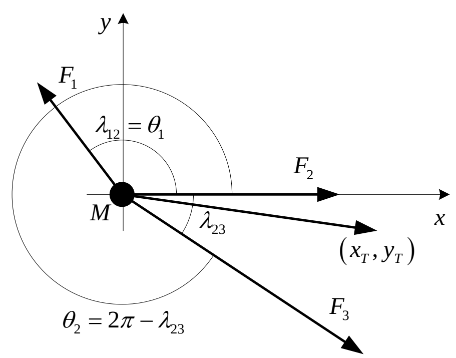
Substituting the given values

$$P_3 = \frac{0.9}{(1.3)(1.4)} - 1 = \frac{0.9}{1.82} - 1 = -0.51 = -51\%$$ [20%]

Problem 3. [16 points]

Given the co-planar forces F_i, i = 1,2,3 separated by angles λ_{12} and λ_{23} that are concurrently applied to a point mass of M as shown in the figure, establish a coordinate system and derive the coordinates (x_T,y_T) of the point mass after time interval T.





Solution: The simplest Cartesian coordinate system would be one drawn along one of the given force vectors. From the problem figure, we see that F_2 is already in the 'x-direction', so the following solution will proceed on the basis that this will be the chosen direction of the x-axis with M at the origin. Other orientations will be equally acceptable and follow the development below. The general thrust of the solution will be to first calculate the resultant force vector and apply that to the point mass for time T to calculate that required (x_T,y_T).

The student should first illustrate his/her solution with a figure that looks like the one here that clearly shows the orientation of the coordinate axes, and the CW angles with respect to the +x-axis that the forces make with the axes. [25%]

Next, we resolve all the forces into their normal x-y components, and then add the components in their respective directions. The x direction forces are

$$
\begin{aligned}
F_{x,1} &= F_1 \cos\theta_1 = F_1 \cos\lambda_{12} \\
F_{x,2} &= F_2 \\
F_{x,3} &= F_3 \cos\theta_2 \\
&= F_3 \cos\left(2\pi - \lambda_{23}\right) = F_3 \cos\lambda_{23},
\end{aligned}
$$

and the y direction force are

$$
\begin{aligned}
F_{y,1} &= F_1 \sin\theta_1 = F_1 \sin\lambda_{12} \\
F_{y,2} &= 0 \\
F_{y,3} &= F_3 \sin\theta_2 \\
&= F_3 \sin\left(2\pi - \lambda_{23}\right) = -F_3 \sin\lambda_{23}
\end{aligned}
$$

Using the above components, the resultant forces in the coordinate directions are

$$F_X = \sum_{i=1}^{3} F_{x,i}, \quad F_Y = \sum_{i=1}^{3} F_{y,i}.$$ [25%]

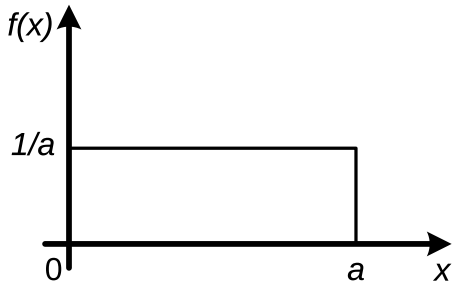
The resultant force is then $F_R = \sqrt{F_X^2 + F_Y^2}$ which is directed at the angle $\theta_R = \tan^{-1}\left(F_Y / F_X\right)$ that is again measured CW from +x-axis.

From elementary mechanics, a stationary point subjected to acceleration a for time T moves the distance $s = aT^2/2$ in the direction of the acceleration vector. In our problem Newton tells us that $a = F_R/M$, therefore $s = \dfrac{F_R T^2}{2M}$. [25%]

Finally, the polar coordinates (s, θ_R) of the point mass M after time T are resolved back into the desired rectangular coordinates in the established coordinate system as

$$
\begin{aligned}
x_T &= s\cos\theta_R \\
y_T &= s\sin\theta_R
\end{aligned}.
$$ [25%]

Problem 4. [14 points]



A probability density function (pdf) like the well known Gaussian or bell curve is used to define the distribution of a random variable (r.v.) like a test score over its range. An even simpler pdf is the uniform distribution or 'boxcar' shown in the figure.

Recall that the area under every pdf must equal unity. The mean, expected value, or average value of a r.v. can be computed from its pdf *f(x)* by the formula

$\bar{x} = \int_{-\infty}^{\infty} x f(x)\,dx$, and a measure of its dispersion, called standard deviation σ_X is computed from its variance, $\sigma_X^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x)\,dx$. Using these definitions, find $\bar{x}$ and σ_X for a r.v. defined by the uniform distribution shown in the figure.

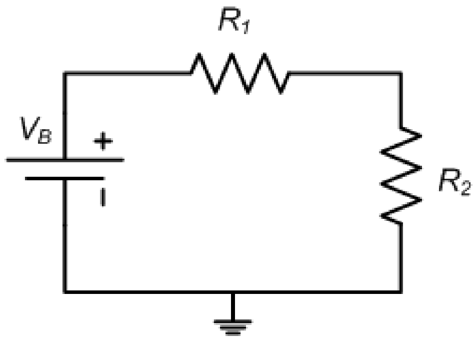
Solution: Substituting *f(x)* from the figure into the formula for the mean yields

$$\bar{x} = \int_0^a \frac{x}{a}\,dx = \left.\frac{x^2}{2a}\right|_0^a = \frac{1}{2}\left[\frac{a^2}{a}\right] = \frac{a}{2}. \qquad [50\%]$$

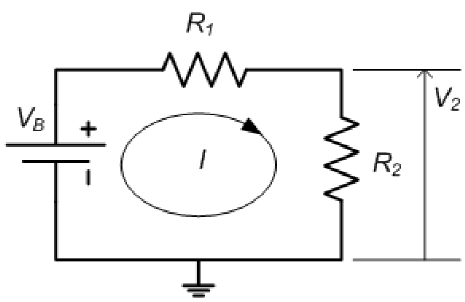
Similarly for the standard deviation, we again substitute the boxcar pdf and use the value for $\bar{x}$ from above to obtain

$$\begin{aligned}
\sigma_X^2 &= \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x)\,dx = \frac{1}{a}\int_0^a \left(x^2 - 2\bar{x}x + \bar{x}^2\right)dx \\
&= \frac{1}{a}\left.\left\{\frac{x^3}{3} - \bar{x}x^2 + \bar{x}^2 x\right\}\right|_0^a = \frac{1}{a}\left(\frac{a^3}{3} - \frac{a^3}{2} + \frac{a^3}{4}\right) \\
&= a^2\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{4}\right) = \frac{a^2}{12}, \quad \therefore \sigma_X = \frac{a}{2\sqrt{3}}
\end{aligned} \qquad [50\%]$$

Problem 5. [12 points]



The figure shows a simple DC circuit containing a battery and two resistors. The values of V_B and R_1 are given. Derive the value of R_2 in terms of R_1 that will maximize the power R_2 will dissipate when placed in the circuit.



Solution: This is the simplified form of the common 'load matching' problem in electrical circuits. The student giving evidence that Ohm's law $V = IR$ and power dissipation $P = IV = I^2R$ are involved counts for [10%]. Drawing the following figure and writing the related voltage and power formulas for R_2 will count for an additional [20%].

$$I = \frac{V_B}{R_1 + R_2},\ V_2 = \frac{R_2}{R_1 + R_2} V_B,\ P_2 = IV_2 = \frac{R_2 V_B^2}{(R_1 + R_2)^2}$$

Now let $R_2 = kR_1$ to express R_2 in terms of R_1 where $k > 0$ is the constant to be determined that will yield the desired value of R_2. Then rewriting the power dissipated in R_2 gives

$$P_2 = \frac{kR_1 V_B^2}{(R_1 + kR_1)^2} = \frac{kR_1 V_B^2}{R_1^2 (1+k)^2} = \frac{k}{(1+k)^2} \frac{V_B^2}{R_1} = \frac{k}{(1+k)^2} C$$

where C is the indicated constant in terms of the known quantities [20%]. Now we find the maximum of P_2 with respect to k in the usual manner using the calculus.

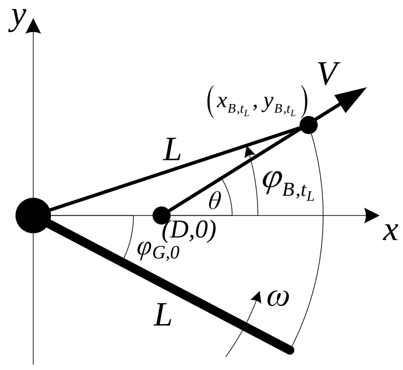
$$P_2 = C \frac{k}{(1+k)^2}$$

$$\frac{dP_2}{dk} = C\left[\frac{1}{(1+k)^2} - \frac{2k}{(1+k)^3}\right] = \frac{C}{(1+k)^2}\left[1 - \frac{2k}{1+k}\right]$$

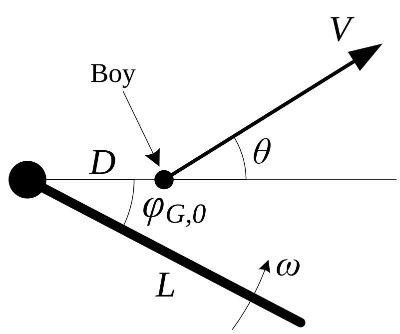
$$\therefore \frac{dP_2}{dk} = 0 \Rightarrow 1 = \frac{2k}{1+k} \rightarrow k = 1 \Rightarrow R_1 \equiv R_2$$

[50%]

Problem 6. [30 points]



A boy sees a large swinging gate coming at him. As shown in the figure above, the gate of length L is approaching him at a constant angular velocity ω (rad/sec), that at time $t_0 = 0$ makes angle $\varphi_{G,0}$ with his position which is a distance $D < L$ from the hinge post. He decides to make a run for it in direction θ, and at time t_0 accelerates 'instantly' to constant velocity V. Set up the dynamic model for the scenario and describe the conditions required for the boy's selected (V, θ) to successfully evade the approaching gate. (Hint: Go as far as you can in setting up and solving this problem. Extra credit is available for 'going above and beyond' what is asked.)



Solution: Concluding that the boy will evade the gate if he can get just beyond radial distance L from the hinge post before the gate hits him is seminal. The student should first indicate his/her understanding by drawing a correct figure [20%].

Referring to the student's figure, the boy's coordinates at any time $t \geq 0$ are $x_B(t) = D + (V\cos\theta)t$ and $y_B(t) = (V\sin\theta)t$. The boy's radial distance ρ_B from the hinge post is then derived from

$$\rho_B^2(t) = \left[D + (V\cos\theta)t\right]^2 + \left[(V\sin\theta)t\right]^2.$$

The time t_L required to reach the point (L, φ_{B,t_L}) is then obtained from solving the triangle

$$\begin{aligned}\rho_B^2(t_L) = L^2 &= D^2 + (2V\cos\theta)t_L + (V^2\cos^2\theta)t_L^2 + (V^2\sin^2\theta)t_L^2 \\ &= D^2 + (2V\cos\theta)t_L + V^2 t_L^2 \\ 0 &= V^2 t_L^2 + (2V\cos\theta)t_L + D^2 - L^2\end{aligned}$$

[20%]

Recognizing the quadratic polynomial, the standard solution is applied to recover the desired t_L. As the solution develops, it is clear from the given parameters that the discriminant is positive. Also, since we require that $t_L \geq 0$, the positive sign must obtain, thereby giving one unique and real feasible solution.

$$\begin{aligned} t_L &= \frac{-2DV\cos\theta \pm \left[\left(2DV\cos\theta\right)^2 - 4V^2\left(D^2 - L^2\right)\right]^{1/2}}{2V^2} \\ &= -\frac{D\cos\theta}{V} \pm \frac{1}{V^2}\left[D^2V^2\cos^2\theta - V^2D^2 + V^2L^2\right]^{1/2} \\ &= -\frac{D\cos\theta}{V} \pm \frac{1}{V}\left[L^2 - D^2\left(1-\cos^2\theta\right)\right]^{1/2} \\ t_L &= \frac{1}{V}\left\{\left[L^2 - D^2\sin^2\theta\right]^{1/2} - D\cos\theta\right\} \geq 0 \end{aligned}$$

[20%]

Using this solution, the boy's coordinates at t_L are then $x_B(t_L) = D + (V\cos\theta)t_L$ and $y_B(t_L) = (V\sin\theta)t_L$. This lets us compute his 'hinge post angle' as $\varphi_B(t_L) = \tan^{-1}\left[y_B(t_L)/x_B(t_L)\right]$. It is this angle that must be greater than the gate's hinge post angle at t_L, which is simply $\varphi_G(t_L) = -\varphi_{G,0} + \omega t_L$. So condition for the boy's escape is seen from the figure to be

$$\tan^{-1}\left[y_B(t_L)/x_B(t_L)\right] = \varphi_B(t_L) \geq \varphi_G(t_L) = -\varphi_{G,0} + \omega t_L .$$

[20%]

Since t_L is a function of the tuplet (V, θ), the so-called control or decision vector, this permits evaluation of the success/adequacy of the chosen tuplet.

Extra credit. Award up to an additional 10% to the extent that the student gives evidence that (numerically) solving the last (transcendental) equation for t_L while varying V and θ will allow the boy to compute the optimal policies for 1) minimum t_L within, say, a constrained V by varying θ, or 2) a global maximum t_L that can be achieved with what smallest V and proper choice of θ. In short, the minimum and maximum successful t_L solutions.

Problem 7. [7 points]

John is older than Sally. A) Given John's and Sally's ages, derive the formula that calculates the number of years until Sally will be *X* times as old as John [40%]. B) Derive the formula to calculate how many years ago was John *Y* times as old as Sally [40%]. For both parts clearly state the constraints on all parameters. C) Using the following values for the ages and the constants, along with your derivations, calculate the answers for parts A and B. $A_J = 31$, $A_S = 20$, $X = 0.9$, $Y = 3$ [20%].

Solution: A) Let N_X be the years until $A_S + N_X = X\left(A_J + N_X\right)$. Solving for N_X gives

$$A_S - XA_J = \left(X-1\right)N_X$$
$$N_X = \frac{A_S - XA_J}{X-1} \geq 0$$

It is clear that $A_S/A_J \leq X < 1$ since the smallest value of X can never be less than A_S/A_J. And since $A_J > A_S$, then A_S/A_J must always be less than one. [40%]

B) Let N_Y be the years until $A_J - N_Y = Y\left(A_S - N_Y\right)$. Solving for N_Y gives

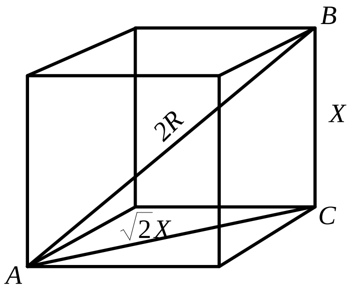
$$YA_S - A_J = \left(Y-1\right)N_Y$$
$$A_S \geq N_Y = \frac{YA_S - A_J}{Y-1} \geq 0$$

Furthermore, we see that $1 < A_J/A_S \leq Y$. [40%]

C) Substituting the given numerical values for part A gives $N_{0.9} = \frac{20-\left(0.9\right)\left(31\right)}{0.9-1} = \frac{-7.9}{-0.1} = 79$, at which time their ages will be $A_J = 110$, $A_S = 99$. Similarly, substituting for part B gives $N_3 = \frac{3\left(20\right)-31}{3-1} = \frac{29}{2} = 14.5$, at which time their ages were $A_J = 16.5$, $A_S = 5.5$. [20%]

Problem 8. [10 points]

What fraction of sphere's volume will be displaced by the largest cube that the sphere can enclose?



Solution: The problem is solved by first realizing that the largest cube that fits inside a sphere is one with its eight corners touching (osculating) the sphere's surface. Drawing a correct picture of this is critical to proceeding with the solution [20%].

Then the student must recognize that the line connecting the diagonally opposite corners of the cube passes through the center of the sphere of radius *R* and is of length 2*R* [20%]. The remainder of the problem requires using elementary geometry to derive the volume of the cube as a function of *R* [35%]. And finally the cube's volume must be divided by the volume of the sphere to obtain the required fraction [25%].

Line *AB* of length 2*R* passes through the center of the sphere and connects the opposite diagonals of the cube with its endpoints on the surface of the sphere. Line *AC* is the familiar diagonal of a square. The cube with side length *X* then forms the right triangle *ABC* as shown. The development below follows from the above geometry.

$$(2R)^2 = X^2 + \left(\sqrt{2}X\right)^2 = 3X^2$$

$$X^2 = \frac{4R^2}{3} \rightarrow X = \frac{2R}{\sqrt{3}}$$

The cube's volume $V_C = X^3$, and the sphere's volume $V_S = \frac{4}{3}\pi R^3$. Substituting and forming the required ratio and working out the arithmetic gives

$$\frac{V_C}{V_S} = \frac{\dfrac{8R^3}{3^{3/2}}}{\dfrac{4\pi R^3}{3}} = \frac{2}{\sqrt{3}\pi} = 0.3676\,,$$

a somewhat surprisingly small number.

