TechTest2011

Merit Scholarship Examination
in the Sciences and Mathematics
given on 26 March 2011

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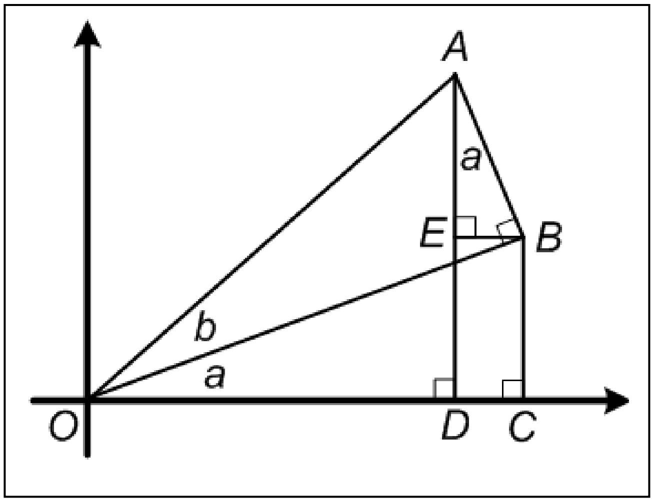
The Sierra Environmental Studies Foundation

Solutions Key

Problem 1 (6 points): A Double Angle Trig Proof – Prove that sinθ = 2sin(θ/2)cos(θ/2). Hint: The road there goes through sin(a+b); draw the figure and sally forth. This is a freebie that you might need later.

Solution to Problem 1: Requires student to first prove that sin(a+b) = sin(a)cos(b) + sin(b)cos(a). From the figure we can write the following relations by inspection.



$$\sin(a+b) = \frac{AD}{AO} = \frac{AE+ED}{AO} = \frac{AE+BC}{AO}$$

$$\sin a = \frac{BC}{BO} \rightarrow BC = BO\sin a = ED,$$

$$\cos a = \frac{AE}{AB} \rightarrow AE = AB\cos a.$$

$$\sin b = \frac{AB}{AO} \rightarrow AB = AO\sin b, \quad \cos b = \frac{BO}{AO} \rightarrow BO = AO\cos b.$$

$$\sin(a+b) = \frac{AB\cos a + BO\sin a}{AO} = \frac{AO\sin b\cos a + AO\cos b\sin a}{AO}$$

$$= \sin a\cos b + \sin b\cos a.$$

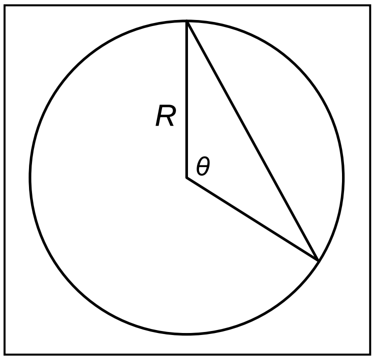
[50%]

Now let a = b = θ/2, and substitute in the above to complete the required proof.

$$\sin(a+b) = \sin a\cos b + \sin b\cos a \rightarrow \sin\left(\frac{\theta}{2}+\frac{\theta}{2}\right) = \sin\frac{\theta}{2}\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

$$\therefore \quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}, \quad \text{QED}$$

[50%]



Problem 2 (10 points): Sectors and Segments of a Circle – The sector of a circle is the region between two distinct radii and their intercepted arc. The segment of a circle is the region between a chord and its associated arc. (A, 20%) Derive the formula for the area of a sector whose defining radii of length R are separated by angle θ. (B, 80%) Derive the area of a segment whose defining arc subtends angle θ at the center of a circle of radius R.

Solution to Problem 2: (A) The area of a sector $A_{\text{sector}}(\theta)$ is proportional to the area of the circle in the proportion of its defining angle θ radians to 2π, the number of radians in a circle. Therefore

$$A_{\text{sector}}(\theta) = \frac{\theta}{2\pi}\pi R^2 = \frac{\theta R^2}{2}$$ [20%]

(B) By inspection, the area of the indicated segment is simply the area of the related sector, as shown in the figure, minus the area of the isosceles triangle of altitude $R\cos(\theta/2)$ and base $2R\sin(\theta/2)$. Therefore the area of the triangle is $(R\cos(\theta/2))(2R\sin(\theta/2))/2 = R^2\sin(\theta/2)\cos(\theta/2) = R^2\sin(\theta)/2$. Finally

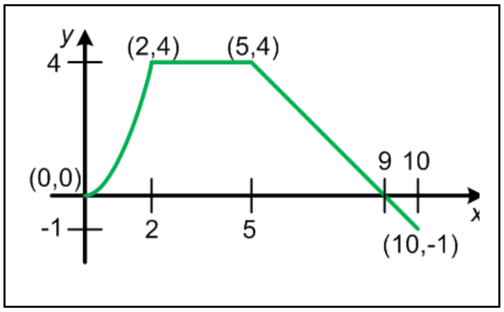
$$A_{\text{segment}}(\theta) = \frac{\theta R^2}{2} - \frac{R^2\sin\theta}{2} = \frac{R^2}{2}(\theta - \sin\theta).$$ [80%]

Problem 3 (10 points): Using Logicals to Express Complex Formulas – Suppose you wanted to express and compute a function $f(x)$ that equaled another function $g(x)$ for all $x < 5$, and then equaled the sum of the two functions $g(x) + h(x)$ for all $x \geq 5$. Using logicals which evaluate to 0 or 1 allows a straightforward way to express, program, and even calculate in a spreadsheet such arbitrarily complex functions. The above example would simply be written $f(x) = g(x) + (x \geq 5)h(x)$, where $(x \geq 5)$ is the logical that evaluates to unity when its argument $x \geq 5$ is TRUE and zero when its argument is FALSE. The argument can be any logical expression such as <, >, =, &, OR, ¬(not), and so on.

Using the above explanation, plot the following function over the range $0 \leq x \leq 10$ labeling all necessary points to demonstrate your understanding of how to use logicals in expressions and formulas.

$$y = (x \leq 2)x^2 + (x > 2)(x \leq 5)4 + (x > 5)(9 - x).$$

Solution to Problem 3:

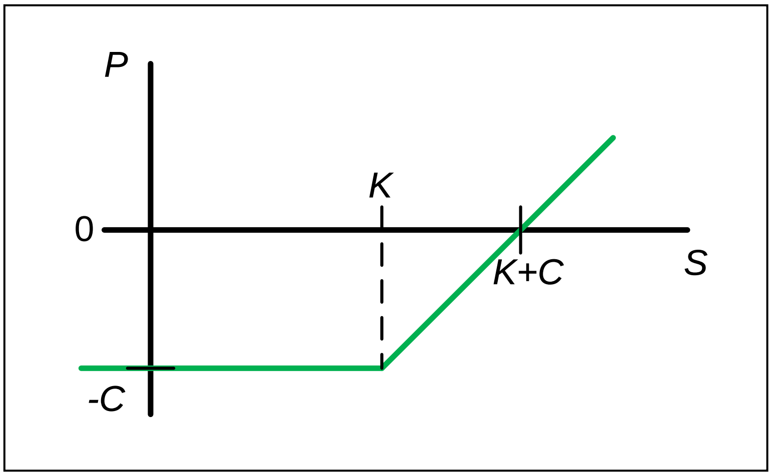


Problem 4 (15 points): Stock Option Basics – Understanding options is basic to all areas of commerce. In trading stocks the 'call option' gives its purchaser the right, but not the obligation, to buy a share of its underlying stock at $*K*/share (the strike price) before the option expires (the expiry date). If the stock's share price is $*S*, then the option holder stands to collect an amount $(S\text{-}K)(S>K)$ when the option is exercised. If the share price never exceeds the option's strike price before expiry, then the option expires worthless and the investor loses $*C*, the cost of the option. Using what you learned in Problem 2, answer the following questions.

(A, 10%) If the option costs C to purchase, write the expression for the profit $*P* as a function of S from buying and exercising this option. (Hint: Recall that profit equals income in excess of costs.)
(B, 40%) Draw a plot of P as a function of S and label all critical points.
(C, 50%) If the current price of the stock is S_0, and you buy a call option with $K = 1.05S_0$ that costs $C = 0.04S_0$, what percent profit can the option buyer make if the stock appreciates a maximum of 15% before expiry.

Solution to Problem 4: (A) $P = (S\text{-}K)(S>K) - C$. [10%]

(B) [40%]

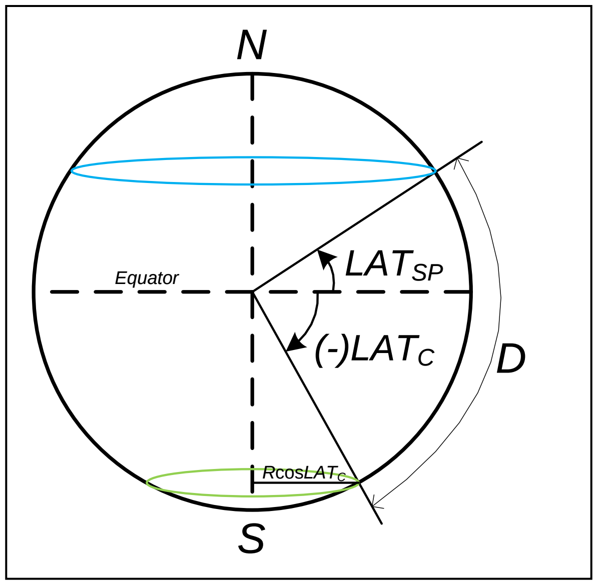


(C) Given S_0, $K = 1.05S_0$, $C_0 = 0.04S_0$, $S_{\max} = 1.15S_0$, we see that $S_{\max} > K$. Since then we can exercise the option and purchase a share for K, our profit from part (A) is $S_{\max} - K - C$ on an investment of C. The simple profit percent calculation then follows immediately.

$$\% \text{ profit } = \frac{S_{\max} - K - C}{C} = \frac{(1.15 - 1.05 - 0.04)S_0}{0.04S_0} = \frac{0.06}{0.04} = 1.5 = 150\%. \qquad [50\%]$$

Problem 5 (17 points): Closed Tour on Spherical Globe – On a spherical globe the rule for traversing a closed tour (path) with three equal length legs is first go due South, then due East, then due North until you arrive at your starting point. The North Pole is a point where such a tour may be started. Are there any other starting points where such tours may be executed? If so, then describe and parametize them; if not, state your proof.

Solution to Problem 5: Yes, there are an infinite number of such starting points, besides the North Pole, that satisfy the requirements for such a tour. All of them have coincident first and third legs, with the second leg consisting of one or more constant latitude circum-navigations of the globe. Using the standard LAT/LONG convention, we draw the geometry as shown in the figure (recall, southern latitudes are of negative sign). [20%]



Let R be the radius of our spherical globe. Then an infinite number of starting points (SPs) may be placed on a latitude LAT_{SP} (blue) from which we may travel to a southern latitude LAT_C at which a constant latitude circumnavigation (green) is equal to the distance $D = (LAT_{SP} - LAT_C)R$ where the latitude angles are expressed in radians. [20%]

This requires that radian angles LAT_{SP} and LAT_C be in the open interval (π/2, -π/2), which is equivalent to (90^0, -90^0), and satisfy

$$(LAT_{SP} - LAT_C)R = n(2\pi R \cos LAT_C) \text{ or,}$$
$$LAT_{SP} = 2\pi n \cos LAT_C - LAT_C, \quad n = 1, 2, 3, \ldots$$

where n is the number of constant latitude circumnavigations. So the choice of any allowable LAT_C and n pair that calculates an allowable LAT_{SP} will therefore specify an infinite number of SPs on that starting latitude each having a different longitude. [60%]

(The advanced student who goes 'above and beyond' and points out that 1) feasible LAT_C values are only south of the equator, and/or 2) that this family of solutions may also include the North Pole with a discussion of the latitudes at which that starting point will also yield an infinity of solutions. The latter is beyond the scope of the problem as stated, but given that s/he satisfactorily completed the previous parts, up to an extra 20% of the problem's point value may be awarded for parts 1) and 2).)

Problem 6 (19 points): Reverse Osmosis Efficiency - A water desalinization plant is currently using E amount of energy to produce a volume V of potable water. The plant then installs new reverse osmosis (RO) desalinization technology and finds that it can now produce 70% more water for the same energy. Suppose that in pre-RO operations desalinization contribute 60% of the cost of operating the plant. (A, 50%) In the new configuration, what percentage cost savings will RO provide? (B, 50%) What level of energy efficiency for water production would be required if the plant wants to cut its total operating costs by 50%? Assume the cost of energy is proportional to the energy used, and the volume of water produced pre- and post-RO is proportional to the energy used.

Solution to Problem 6: (A) Energy saved with RO to produce water volume V. First set up the known relationships with proportionality constants k and l as

$$V = kE,\;\; 1.7V = k_{RO}E \Rightarrow V = k_{RO}\left(\frac{E}{1.7}\right) = k_{RO}E_{RO}$$

Then costs are $C_E = lE = 0.6C_{Tot},\;\; C_{E,RO} = lE_{RO} = l\left(\frac{E}{1.7}\right),$

giving cost saving ratio of $\dfrac{C_{E,RO}}{C_E} = \dfrac{l\left(\frac{E}{1.7}\right)}{lE} = \dfrac{1}{1.7} = 0.5882.$

So 60% of costs were reduced by (0.5882)(0.6) = 0.3529 of pre-RO costs. The post-RO total costs can then be calculated as

$$C_{Tot,RO} = \left(0.3529 + 0.4\right)C_{Tot} = 0.7529C_{Tot}.$$

This provides a reduction in the plant operation's total cost of (1 - 0.7529) = 0.2471 or 24.71%. (50%)

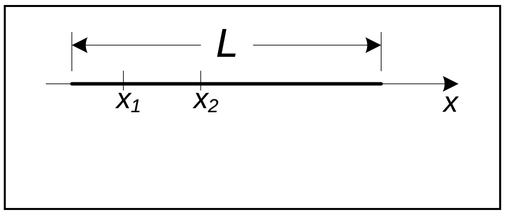
(B) For a 50% reduction in C_{Tot} we require that $C_{Tot,RO} = 0.5C_{Tot}$. From part (A) the new post-RO total must then be expressed as $C_{Tot,RO} = (0.1+0.4)C_{Tot}$. Then working backward, again using the development of part (A), we must have

$$0.1 = \left(\frac{C_{E,RO}}{C_E}\right)0.6 \rightarrow C_{E,RO} = \left(\frac{0.1}{0.6}\right)C_E,$$

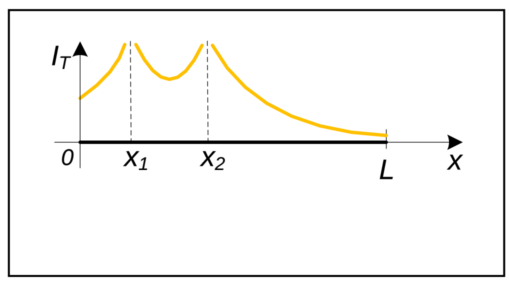
or the RO technology must be six times as energy efficient as the pre-RO technology. Another way of stating this is in terms of production increase P_{RO}. Then

$$\frac{C_{E,RO}}{C_E} = \left(\frac{1}{1+P_{RO}}\right) = \frac{1}{6} \rightarrow P_{RO} = 6 - 1 = 5 = 500\% .$$

This says that a newer RO technology must increase produced volume V_{RO} by 6 times or 500% to reduce the plant's total operating costs by 50%. (accept either method, 50%)



Problem 7 (23 points): Maximal Illumination of Path – Where would you place two identical lights to maximally illuminate a straight line path of length L? For this problem assume that the intensity at a distance $d > 0$ from each light is proportional to the inverse square law, or $I = k/d^2$ where k is a constant (note that I has a singularity at $d = 0$). The total illumination I_T at any point along L is the sum of the intensity contributions from each light. By 'maximally illuminate' we mean to maximize the minimum level of illumination along L. Your derived solution should specify the values of the coordinates x_1 and x_2 for the lights (see figure), and you should show that your solution is correct.



Solution to Problem 7: Given the inverse square law for intensities and the above figure, the student should have in mind a plot of total intensity I_T along L that looks something like the figure here. So where to place the lights? Well, if we co-locate them at $L/2$ then the minima will occur at 0 and L. But we can immediately increase one of the minima by moving one light a small distance toward an endpoint. The total intensity at the other endpoint can be equally increased by moving the other light toward that endpoint.

It quickly becomes clear that when the lights are separated, then there will be three minima as shown in the figure – one each at the endpoints, and one at half way between the lights. And to satisfy the 'maximally illuminate' criterion, all three minima must be made equal. This calls for the symmetric placement of the lights along L that can be specified by the selection of only one parameter, the distance s – i.e. the coordinates of the lights will be at $x_1 = s$ and $x_2 = L - s$ giving the three minima as $I_T(0)$, $I_T(L/2)$, $I_T(L)$. From the symmetry we know that any selection of s will yield $I_T(0) = I_T(L)$, therefore the only problem remaining is to find s such that $I_T(0) = I_T(L/2)$. [50%]

We then write the total intensity at x as the sum of the individual intensities produced by the two lights. Therefore

$$I_T(x) = k\left[\frac{1}{(x-s)^2} + \frac{1}{(x-L+s)^2}\right]$$

Then at $x = 0$ and $L/2$ we have the total intensities

$$I_T(0) = k\left[\frac{1}{(s)^2} + \frac{1}{(s-L)^2}\right],\ \ I_T\left(\frac{L}{2}\right) = k\left[\frac{1}{\left(\frac{L}{2}-s\right)^2} + \frac{1}{\left(\frac{L}{2}-L+s\right)^2}\right] = k\left[\frac{2}{\left(s-\frac{L}{2}\right)^2}\right]$$

and $I_T(0) = I_T\left(\frac{L}{2}\right) \Rightarrow \frac{1}{(s)^2} + \frac{1}{(s-L)^2} = \frac{2}{\left(s-\frac{L}{2}\right)^2}$.

From here we slog through some algebra to solve for s.

$$\frac{(s-L)^2+s^2}{s^2(s-L)^2} = \frac{2}{\left(s-\frac{L}{2}\right)^2} \to \left(s^2-2sL+L^2+s^2\right)\left(s^2-sL+\frac{L^2}{4}\right) = 2s^2\left(s^2-2sL+L^2\right)$$

$$0 = \frac{3L^2s^2}{2} - \frac{3L^3s}{2} + \frac{L^4}{4} = 6L^2s^2 - 6L^3s + L^4$$

$$s = \frac{L}{2} \pm \frac{\sqrt{3}}{6}L = L\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) = 0.2113L \ \text{ since we must have } 0 \le s < L/2.$$

To check the answer we let $L = 1$, and then, using our established criterion $I_T(0) = I_T(L/2)$, verify that $I_T(0) = I_T(0.5)$ when $s = 0.2113$.

$$I_T(0) = k\left[\frac{1}{(0.2113)^2} + \frac{1}{(0.2113-1)^2}\right] = 24.0052k$$

$$I_T(0.5) = k\left[\frac{1}{(0.5-0.2113)^2} + \frac{1}{(0.5-1+0.2113)^2}\right] = 23.9959k$$

which is close enough for government work (round off error). [50%]