TechTest2013

Merit Scholarship Examination
in the Sciences and Mathematics
given on 23 March 2013

sponsored by

The Sierra Economics and Science Foundation

Solutions Key

Problem 1: Clock times when hands are coincident (12 points)

The hour and minute hands of a clock coincide exactly several times during a twelve hour cycle. Everyone knows that at 12 o'clock (noon and midnight) the hands are coincident. Calculate all the remaining times to the nearest second when the two hands are again coincident. Hint: the first such time is 1:05:27.

Solution: The hint tells the student that such coincidences occur every 1:05:27 hours, and simple addition of that interval to 12 o'clock and proceeding around the clock dial will yield the required times of which there are ten that define eleven equal intervals between succeeding 12 o'clocks. Each interval is then 1/11 = 0.09… of a circle that consists of 60 minutes, and is therefore equal to 60/11 = 5.45… = 5:27.27… minutes long. Using that measure, the required precision for the answer is obtained from successively adding 1:05:27.27 to 12 o'clock until all the ten required times are computed.

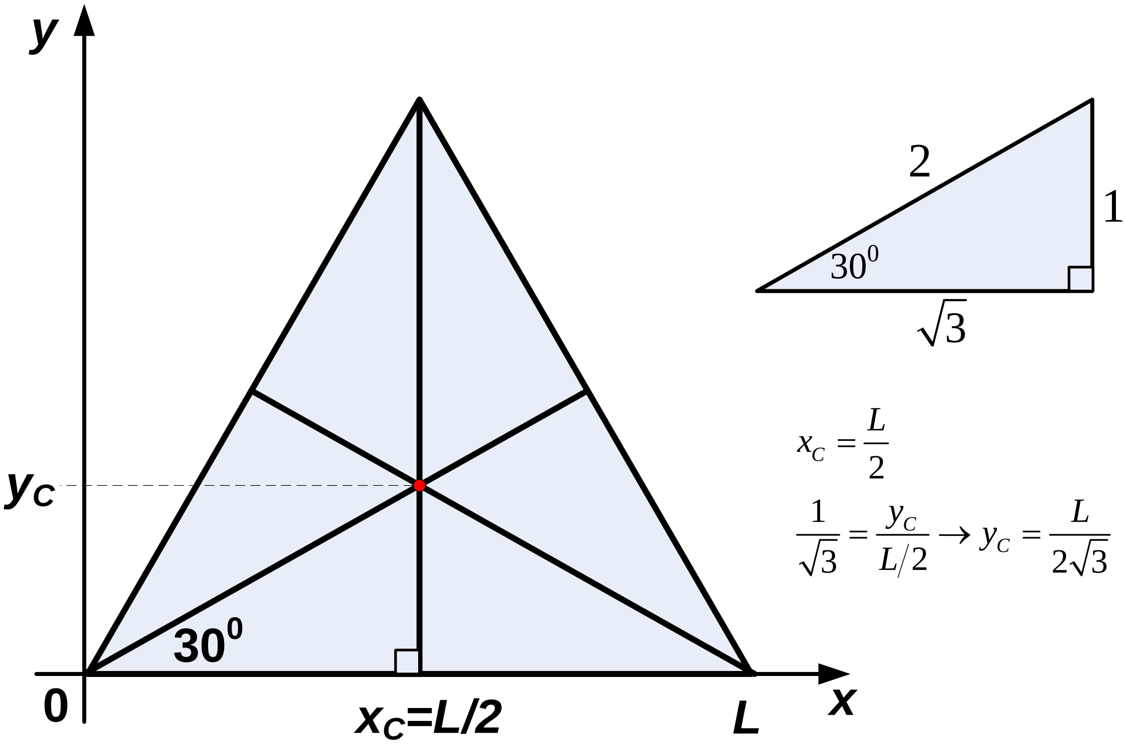
The complete list to the nearest second is then

12:00:00
1:05:27
2:10:55
3:16:22
4:21:49
5:27:16
6:32:44
7:38:11
8:43:38
9:49:05
10:54:33
12:00:00

Problem 2: Coordinates of an equilateral triangle's center (10 points)

Given an equilateral triangle with side length *L*, what are the coordinates of its center? Hint: Place the triangle into a Cartesian coordinate system so that one of its sides lies on the *x*-axis.

Solution: The student should lay out the triangle as shown in the figure, and perform the simple algebra and geometry to calculate the center coordinates as $\left(L/2,\ L/2\sqrt{3}\right)$.

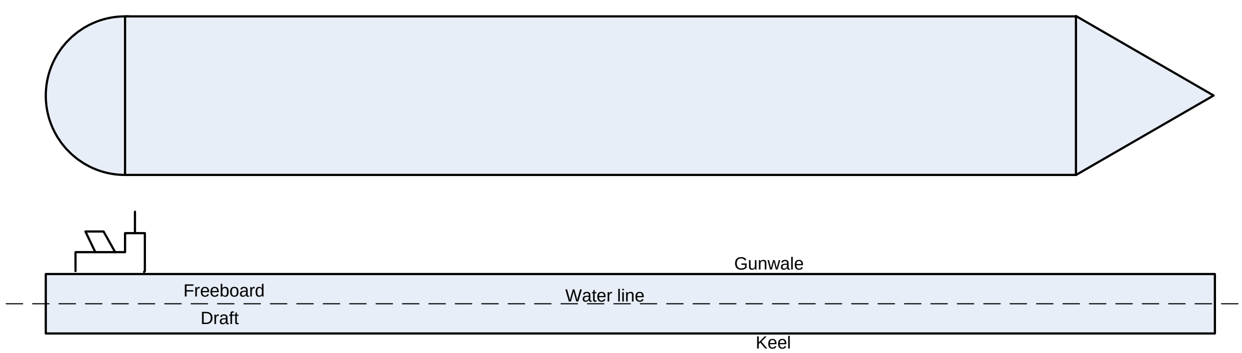


$$x_C = \frac{L}{2}$$

$$\frac{1}{\sqrt{3}} = \frac{y_C}{L/2} \rightarrow y_C = \frac{L}{2\sqrt{3}}$$

Problem 3: The cargo capacity of a supertanker (18 points)

Suppose a supertanker has a very simple ‘cookie cutter’ hull shape with a deck area plan view (from above) that consists of a rectangular center section with an semi-circular stern and a triangular bow as shown in the figure below. For safe sailing the tanker must not be loaded so that its freeboard is less than F_T = 7 meters (m). The empty weight of the ship is M_S = 50K tons, where one ton equals 1,000 kg. If the height of the hull (the keel to gunwale distance) is 30 m, the beam (width) of the ship is 40 m, the length of the bow is 40 m, and the rectangular midsection is 300 m long, then how many tons of crude oil can the tanker safely carry? (Use 1,020 kg/m^3 as the density of sea water, and 900 kg/m^3 as the density of crude oil. Hint: first derive the model for the ship’s freeboard before inserting the numbers for the calculation.)



Solution: Part A (75%) Model Derivation. Let W be the vessel's width or beam; L is the length of midsection; B is the length of the bow; H is the height of the hull that is the sum of its freeboard (F) and draft (D); M_S is the ship's empty mass; M_C is the mass of the cargo (oil); ρ_W is the seawater density; and ρ_C is the cargo's density.

The constant plan view area $A=\frac{\pi}{2}\left(\frac{W}{2}\right)^2+LW+\frac{BW}{2}$, and therefore the volume of the hull is $V = AH$. Recalling that any mass M floats in a fluid when the volume of water it displaces is at least M, requires that the ship's mass is related to its loaded draft by $M_S+M_C=AD_L\rho_W$, the mass of the displaced water, which lets us express the loaded draft as $D_L=\frac{M_S+M_C}{A\rho_W}$. The loaded freeboard is then $F_L = H - D_L$. Safe sailing requires that $F_L \geq F_T$, or $H-D_L=H-\frac{M_S+M_C}{A\rho_W}\geq F_T$. Solving this for the M_C gives the required answer for the maximum cargo of crude oil the tanker may safely transport.

$$M_C \leq HA\rho_W - M_S - F_T A\rho_W = \left(H-F_T\right)A\rho_W - M_S \ .$$

Part B (25%) Numerical solution. Using the values given in the problem statement, the deck area computes to

$$A=\frac{\pi}{2}\left(\frac{W}{2}\right)^2+LW+\frac{BW}{2}=\frac{\pi}{2}\left(\frac{40}{2}\right)^2+300*40+\frac{40*40}{2}=13,428\text{m}^2 \ ,$$

which then gives us the desired answer as

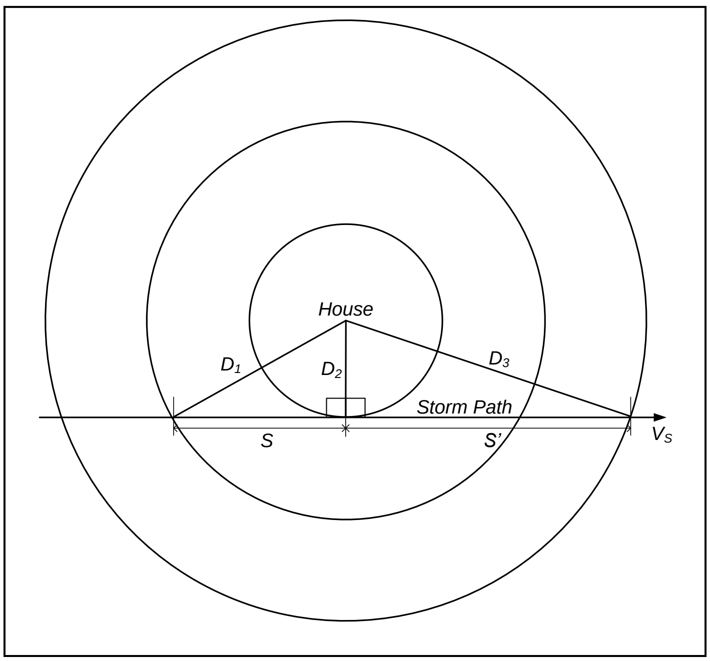
$$M_C \leq \left(H-F_T\right)A\rho_W - M_S = \frac{\left(30-7\right)*13,428*1,020}{1,000}-50,000=265,028 \text{ tons} \ .$$

(We note that there was no need to use the density of crude oil to solve this problem.)

Problem 4: Predicting wind speed of a passing storm (16 points)

On the radio you hear that a fast moving hurricane is approaching. The meteorologist reports that it will be travelling on a straight path at a constant speed for the next 24 hours. She also tells you that the wind speed falls off exponentially as 120*exp(-0.00693*D), where D is in miles from the storm's center. You are concerned about wind damage to your property and have an anemometer which measures only wind speed. You look at your watch, record the anemometer's reading as 40 mph, and notice the wind speed is increasing. Two hours later you see that the wind speed appears to have peaked at 60 mph, and is now decreasing. You don't want to leave your house for an appointment until the wind speed has fallen to 20 mph. How long must you wait? (Hint: what is the shape of a line that is equidistant from a point?) First derive the wait time model before doing any numerical calculations. (Part A: Model Derivation, 75%; Part B: Numerical Calculation, 25%)

Solution: The student should draw three concentric circles centered on his house. The intermediate one at a radius of D_1 is the locus of points where the storm was two hours ago, and the inner one at a radius D_2 is the locus of points where the storm is now at its closest point of approach given the formula for the wind speed. From there the development proceeds in a straightforward manner by drawing any straight line tangent to the inner circle which indicates the constant velocity path of the storm. Working the right triangles yields *S*, the storm's progress over the last two hours and let's us calculate the storm's speed V_S. Finally, D_3, the radius of the third and largest circle for when wind speed will have diminished to the desired 20 mph, gives the distance *S'* that the storm has yet to travel, and thereby yields the required wait time *T'*.



$$V_W = 120e^{-kD} \rightarrow D = -\frac{1}{k}\ln\frac{V_W}{120},$$

$$k = 0.00693$$

$$S = \sqrt{D_1^2 - D_2^2} \rightarrow V_S = \frac{S}{2},$$

$$S' = \sqrt{D_3^2 - D_2^2}$$

$$\text{Wait time (hrs) } T' = \frac{S'}{V_S} = \frac{2S'}{S}$$

(Part A: 75%)

D_1 = 158.5 miles, D_2 = 100.0 miles, D_3 = 258.5 miles, *S* = 123.0 miles, *S'* = 238.4 miles; this yields a wait time of *T'* = 3.88 hrs = 3:53 hrs.
(Part B: 25%)

Problem 5: Collinear Points (10 points)

Given points (3, 9.5), (7, 19.5), (-1, -1.5), (-5, -10.5) in a Cartesian plane, determine whether or not they are collinear. Prove your answer.

Solution: Collinear points in a Cartesian plane all satisfy the same $y = mx + b$ equation of a straight line. That means that they have the same slope *m* and the same *y*-intercept *b*. The slope between any two collinear points (x_1, y_1), (x_2, y_2) is $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. Starting anywhere in the given list of points – e.g. the first – we calculate the pairwise slopes as *m* = 2.5, 2.75, 2.5, and quickly discover that (-1, -1.5) has a different slope, and therefore cannot be collinear with the other three. The y-intercept of the remaining three pairs of points is computed from the familiar point-slope formula

$$m = \frac{y - y_i}{x - x_i} \to y = mx - mx_i + y_i \to b = y_i - mx_i \ .$$

Computing b for the three surviving points (3, 9.5), (7, 19.5), (-5, -10.5), we get $b = 2$ for all of them, therefore confirming that they are collinear.

Problem 6: Differentiation (14 points)

Given $y = \frac{x^2 + \ln x}{(\sin 3x + 4)^2}$, find $\frac{dy}{dx}$. (Hint: express y as a function of multiple subfunctions of x and use the appropriate derivative formula for that expression in order to simplify your work.)

Solution: The most obvious re-expression of y is $y = u(x)/v(x)$, giving

$$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx} = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx} \right)$$

Substituting and taking the elementary derivatives

$$\frac{du}{dx} = \frac{d}{dx}\left(x^2 + \ln x\right) = 2x + \frac{1}{x}, \quad \frac{dv}{dx} = \frac{d}{dx}(\sin 3x + 4)^2 = 2(\sin 3x + 4)(3\cos 3x)$$

$$\text{then } \frac{dy}{dx} = \frac{1}{(\sin 3x + 4)^4}\left\{ (\sin 3x + 4)^2 \left(2x + \frac{1}{x} \right) - 6\left(x^2 + \ln x\right)(\sin 3x + 4)(\cos 3x) \right\}.$$

Performing the algebra and collecting terms give the final answer.

$$\frac{dy}{dx} = \frac{\left(2x + \frac{1}{x} \right)}{(\sin 3x + 4)^2} - \frac{6\left(x^2 + \ln x\right)(\cos 3x)}{(\sin 3x + 4)^3}.$$

Problem 7: Prove a derivative (20 points)

Prove that $\frac{d}{dx}\ln x = \frac{1}{x}$. Hint: Start with the formal definition of a derivative and recall that

$$e^a = \lim_{n\to\infty}\left(1 + \frac{a}{n} \right)^n = \lim_{n\to 0}(1 + an)^{\frac{1}{n}} .$$

Solution:

$$\frac{d}{dx}\ln x = \lim_{\Delta x \to 0}\frac{\ln\left(x+\Delta x\right)-\ln x}{\Delta x} = \lim_{\Delta x \to 0}\left\{\frac{1}{\Delta x}\ln\left(\frac{x+\Delta x}{x}\right)\right\} = \lim_{\Delta x \to 0}\left\{\ln\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}\right\}$$

$$= \ln\left\{\lim_{\Delta x \to 0}\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}\right\} = \ln e^{\frac{1}{x}} = \frac{1}{x}\ln e = \frac{1}{x}\bullet 1 = \frac{1}{x}.\ \text{QED}$$