TechTest2014

Merit Scholarship Examination
in the Sciences and Mathematics
given on 5 April 2014, and

sponsored by

The Sierra Economics and Science Foundation

Solutions Key

Problem 1: Trigonometric Equation (5 points)

Solve sin x + cos x = 1.

Solution: Rewrite equation as sin x = 1 - cos x, and square both sides.

$$\begin{aligned} \sin^2 x &= 1 - 2\cos x + \cos^2 x, \\ 1 - \cos^2 x &= 1 - 2\cos x + \cos^2 x, \\ 2\cos^2 x - 2\cos x &= 2\cos x(\cos x - 1) = 0. \end{aligned} \qquad (50\%)$$

Then from cos x = 0, x = π/2, 3π/2. From cos x = 1, x = 0. Checking these solutions;

For x = 0, sin x + cos x = 0 + 1 = 1;
for x = π/2, sin x + cos x = 1 + 0 = 1;
for x = 3π/2, sin x + cos x = -1 + 0 = -1.

Therefore the required solutions are x = 0, π/2. The solution x = 3π/2 is extraneous in that it arises from the squaring of both sides which does not satisfy the equation to be solved. (50%)

Problem 2: Differentiation (5 points)

Given $y = \dfrac{e^{ax}}{1-\sin^2 x}$, find $\dfrac{dy}{dx}$. (Hint: express y as a function of subfunctions of x and use the appropriate derivative formula for that expression to simplify your work.)

Solution: The most obvious re-expression of y is $y = u(x)/v(x)$, giving

$$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx} = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right) \qquad (33\%)$$

Substituting and taking the elementary derivatives

$$\begin{aligned} &\frac{du}{dx} = \frac{d}{dx}\left(e^{ax}\right) = ae^{ax}, \quad \frac{dv}{dx} = \frac{d}{dx}\left(1-\sin^2 x\right) = \frac{d}{dx}\left(\cos^2 x\right) = 2(\cos x)(-\sin x), \\ \text{then } &\frac{dy}{dx} = \frac{1}{\cos^4 x}\left\{ae^{ax}\cos^2 x - 2e^{ax}\sin x\cos x\right\}, \\ &\frac{dy}{dx} = \frac{e^{ax}}{\cos^2 x}\left\{a - \frac{2\sin x\cos x}{\cos^2 x}\right\} = \frac{e^{ax}}{\cos^2 x}\left\{a - 2\tan x\right\}. \end{aligned} \qquad (67\%)$$

3: Integration (15 points)

It is easier to evaluate some definite integrals by using substitution to change the integration variable, and then, of course, its commensurate limits. Consider the requirement to compute $\int_{y_1}^{y_2} f(y)\,dy = \int_{\Phi(a)}^{\Phi(b)} f(y)\,dy$ where now $y = \Phi(t)$. We presume that a simpler integration problem would be to solve $\int_a^b g(t)\,dt$ such that $g(t) = f(y)\frac{dy}{dt} = f[\Phi(t)]\frac{dy}{dt}$, noticing that substituting the new integrand $g(t)$ into the integral would in effect 'cancel out' the dt, thereby computationally leaving the required dy of the original integral intact even though the new variable of integration is now t. The new limits a and b would be obtained from solving $y_1, y_2 = \Phi(t)$. This approach also lets us integrate mixed function integrals such as $\int_{y_1}^{y_2} h(t)\,dy$ where again $y = \Phi(t)$, which we now ask you to do in this problem. (Hint: find dy and substitute in the integral.)

Let $y = 3t^2 - t$, and calculate $z = \int_0^4 (4t+1)\,dy$.

Solution: First, we reformulate the integrand and the integral's limits in terms of the single variable t.

$$\begin{aligned} dy &= (6t-1)\,dt, \\ y(t_0) &= 0 = 3t_0^2 - t_0 = t_0(3t_0 - 1) \rightarrow\ t_0 = 0, \tfrac{1}{3}, \\ y(t_1) &= 4 = 3t_1^2 - t_1 \rightarrow 3t_1^2 - t_1 - 4 = 0 \rightarrow t_1 = \frac{1 \pm 7}{6} = \tfrac{4}{3}, -1. \end{aligned}$$

(30%)

The integral then becomes $z = \int_0^4 (4t+1)\,dy = \int_{t_0}^{t_1} (4t+1)(6t-1)\,dt$. Working this out yields the four solutions to the integral for the two sets of upper and lower limits. (50%)

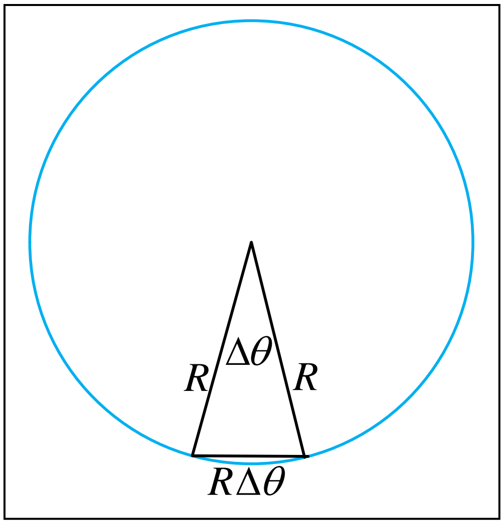
$$z(t_0, t_1) = \int_{t_0}^{t_1} (24t^2 + 2t - 1)\,dt = \left[8t^3 + t^2 - t\right]_{t_0}^{t_1}$$

Then

$$\begin{aligned} z\left(0, \tfrac{4}{3}\right) &= \left[8t^3 + t^2 - t\right]_{t_0}^{t_1} = 8\left(\frac{64}{27}\right) + \frac{16}{9} - \frac{4}{3} - 0 = 15\frac{5}{9} = 15.\overline{55}, \\ z(0, -1) &= (-8 + 1 - 1) - 0 = -8.0, \\ z\left(\tfrac{1}{3}, \tfrac{4}{3}\right) &= 15\frac{5}{9} - \left[8\left(\frac{1}{27}\right) + \frac{1}{9} - \frac{1}{3}\right] = 15\frac{13}{27} = 15.\overline{481}, \\ z\left(\tfrac{1}{3}, -1\right) &= -8 - \left[8\left(\frac{1}{27}\right) + \frac{1}{9} - \frac{1}{3}\right] = -8\frac{2}{27} = -8.\overline{074}. \end{aligned}$$

(20%)

(We note that were such an integral encountered in a realworld problem, then the solution set would most certainly be pruned by the problem's physical and/or temporal constraints.)



Problem 4: Area of a Circle (10 points)

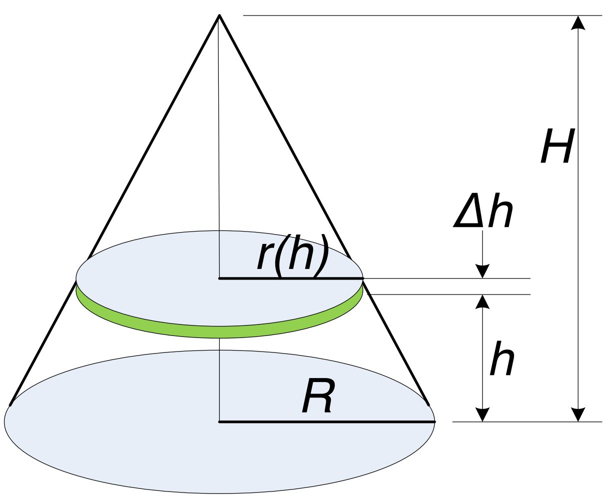
Derive the formula for the area of a circle of radius R given that its circumference $C = 2\pi R$.

Solution: Student should draw picture to indicate understanding that the area of a circle is the sum of small isosceles triangles that becomes the integral of infinitesimal triangles as the apex angle $\Delta\theta$ is taken to the limit. (25%)

The approximate area of the indicated triangle is $\Delta A(\Delta\theta) \simeq \frac{R^2\Delta\theta}{2}$, since for small $\Delta\theta$ the base and altitude of the triangle are approximately $R\Delta\theta$ and R, and become the correct values in the limit. The concept here is to sum the small triangles to 'fill' the circle by letting $\Delta\theta$ approach zero in the limit. This is recognized as the integral over θ when the infinitesimal triangles of area $\frac{R^2 d\theta}{2}$ fill the entire central angle of a circle from 0 to 2π radians. Performing the integration then gives the desired answer.

$$A = \lim_{\Delta\theta\to 0}\sum_i \Delta A_i(\Delta\theta) = \int_0^{2\pi}\frac{R^2 d\theta}{2}$$
$$= \pi R^2$$

(75%)



Problem 5: Volume of a Right Cone (10 points)

Derive the formula for the volume of a right cone of base radius R and height H.

Solution: Again the student should draw the correct figure, and indicate that the cone's volume is obtained in the limit from the sum of the volumes of circular discs of height Δh and decreasing radii which are a linear function $r(h)$ of their perpendicular distance h from the base. (35%)

$$r(h) = R\left(1 - \frac{h}{H}\right)$$

$$V = \lim_{\Delta h \to 0} \sum_{i=1}^{H/\Delta h} \pi r^2(h_i)\Delta h = \pi \int_0^H r^2(h)\,dh$$

$$= \pi R^2 \int_0^H \left(1 - \frac{h}{H}\right)^2 dh$$

$$= \pi R^2 \int_0^H \left(1 - \frac{2h}{H} + \frac{h^2}{H^2}\right) dh$$

$$= \pi R^2 \left(H - H - \frac{H}{3}\right) = \frac{\pi R^2 H}{3}$$

(65%)

Problem 6: The Price of a Bond (10 points)

When issued, a long term Treasury bond that yields (over its life) an annual coupon amount of C is sold for price P. This says that at issuance the bond has an interest 'percent' yield of $I = C/P$. As interest rates go up and down, the market price of the bond will change so that its coupon amount reflects the current market's interest rate. A, 40%) If at a future time this interest rate becomes $I + \Delta I$, then what is the new market price of the bond? B, 40%) Express the fractional (percent) change in a bond's price in terms of the fractional change in the interest rate. C, 20%) If P = \$10K, C = \$500, and I changes from 5% to 6%, what is the percentage change in the market price of the bond?

Solution: A) Let the new/changed price of the bond be $P + \Delta P$. Then the new interest rate that the marketable bond must yield is $I + \Delta I = C/(P + \Delta P)$. The new market price of the bond is then computed simply as $P + \Delta P = C/(I + \Delta I)$. (40%)

B) The fractional change in the interest rate is $\Delta I/I$, and the desired fractional change in the bond's price is $\Delta P/P$. So, using the solution to part A, we want to find

$$\frac{\Delta P}{P} = f\left(\frac{\Delta I}{I}\right)$$

$$\frac{P + \Delta P}{P} = \frac{C}{P}\left(\frac{1}{I + \Delta I}\right) = \frac{I}{I + \Delta I}.$$

$$\frac{\Delta P}{P} = \frac{I}{I + \Delta I} - 1 = \frac{I - I - \Delta I}{I + \Delta I},$$

$$\frac{\Delta P}{P} = \frac{-\Delta I}{I + \Delta I} = -\frac{\Delta I}{I}\left(1 + \frac{\Delta I}{I}\right)^{-1}$$

From this we see the well-known fact that as interest rates rise, bond prices fall, and vice versa. (40%)

C) Substituting the dollar amounts into the solution to part B, gives I = \$500/\$10K = 0.05, and $\Delta I/I$ = 1%/5% = 0.20. Therefore

$$\frac{\Delta P}{P} = -\frac{\Delta I}{I}\left(1+\frac{\Delta I}{I}\right)^{-1} = -0.20(1+0.20)^{-1} = -0.167 = -16.7\% \qquad (20\%)$$

(This says that the current market price of the face value \$10K bond is now \$8,333.33.)

Problem 7: Galileo climbs the Tower of Pisa (5 points)

Galileo is reputed to have climbed the leaning tower to settle once and for all Aristotle's claim that heavier objects fall to the ground faster. The experiment showed that, neglecting air resistance, both heavier and lighter masses were accelerated equally and hit the ground (almost) concurrently. Now Galileo did not have the benefit of Newton's laws, but you do. Prove that in a vacuum masses m and $10m$ have the same acceleration profile and hit the ground at the same time when attracted by the mass of the earth $m_E >> m$.

Solution: Newton demonstrated that $F = ma$, and that two masses attracted each other with the force of gravity that given by $F = Gm_1m_2/r^2$ where G is the gravitational constant and r the distance separating the masses. So if we calculate the acceleration of m due to the force of gravity, we get

$$ma = \frac{Gmm_E}{r^2} \rightarrow a = \frac{Gm_E}{r^2} \ ,$$

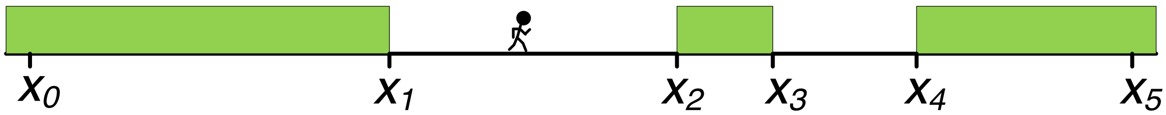
and for the other mass of $10m$ we have

$$10ma = \frac{G10mm_E}{r^2} \rightarrow a = \frac{Gm_E}{r^2} \ ,$$

showing that both masses accelerate equally, therefore gain velocity equally, and therefore hit the ground at the same time.

Problem 8: Where to look, when, and for how long (20 points)

Suppose you are charged to report on the whereabouts of a person who walks the same path every day at the same approximate time. As shown in the (not to scale) figure, her walk takes her back and forth on a circuit between two points, x_0 and x_5, over a straight path on which she is not visible to you the whole time because there are obstructing walls that will not let you see her from your chosen vantage point. She walks at a constant speed of V = 4,500 yards per hour. Her path is $x_5 - x_0$ = 1,500 yards long and defiladed by high walls as shown. She is known to walk at least one hour during her exercise session. Letting x_0 = 0, the coordinates of the other points along the path are x_1 = 500 yds, x_2 = 900 yds, x_3 = 950 yds, x_4 = 1,150 yds, x_5 = 1,500 yds.



Your objective is to ascertain and report her presence or absence as quickly as possible. When you arrive at your observation point, she is nowhere in sight. You note the time (t_0) and start scanning for her whereabouts. Since you cannot look concurrently at all points on the visible portions of her path, your scanning is made up of a sequence of glimpses that cover a more narrow focus and of a duration enough to ascertain with high likelihood that the walker is there or not. Describe a reasonable scanning policy (sequence of glimpses) which will allow you to meet your objective. In short, where will you look (focus your attention) when and for how long? (Hint: Your answer will be a written description with all additional assumptions clearly stated and appropriate numbers included as necessary.)

Solution: This is a purely critical thinking problem with a little arithmetic added. The allocation of credit is here more arbitrary with the main thrust being to verify that the student did have certain critical insights, but not necessarily in the order presented here.

The student should first label and calculate the lengths of the three walls – L_1 = 500 yds, L_2 = 50 yds, L_3 = 350 yds – and index all stated times from the start of scanning. Since the walker is not visible, she must be walking behind one of the three walls and will eventually emerge at points x_1 through x_4, therefore all glimpses should be allocated to those four points. (15%)

Since her position at t_0 can be anywhere behind the walls where she can be walking toward the right or left. And the probability that she is behind any of the walls is proportional to the time it takes her to disappear and then emerge from each of the walls. Since she walks at constant speed, these probabilities should be proportional to the wall lengths divided by 900 yards, the total length of the walls. But is that true since it takes a maximum of $2L_1/V$ = 1000/4500 = 0.222 hr to walk behind wall #1; similarly a maximum of $2L_3/V$ = 700/4500 = 0.156 hr to walk behind wall #3. Wall #2 presents a problem because it has two points of entry and emergence, and therefore for scanning purposes it's as if it is actually two walls with each wall having a maximum dwell time behind the wall of L_2/V = 50/4500 = 0.011 hr.

At this point the student should conclude that if the observer does not see the walker after scanning for 0.222 hr = 13.3 minutes, then he can report that the walker is not on her walk. (30%)

In a similar fashion the observer can eliminate scanning x_2 and x_3 if the walker does not emerge within 0.011 hr = 40 seconds of his arrival. And he can eliminate including x_4 in his scanning after not seeing the walker past the 0.156 hr = 9.36 minute mark of his arrival, after which time all of his glimpses will be focused at x_1. (20%)

Now comes the hard part – how to allocate the glimpses between the four points being observed. Upon arrival, where should he look? If she is still visible he will detect her if he glimpses her before she disappears behind a wall. The biggest penalty to be paid is if she is about to step behind one of the long walls, because missing her then will result in a long delay until she emerges. So it seems that to glimpse first at x_1 and then x_4 would be a reasonable policy before quickly scanning the rest of the visible interval to ascertain her absence. This needs to be traded off against glimpsing at x_3 and x_4 because she will emerge soonest from behind wall #2 given that she was behind it when the observer arrived. But x_3 and x_4 can always be sequenced after eliminating the disappearance behind walls #1 and #3. So the initial glimpse sequence might be x_1, x_4, x_3, x_2.

The remaining glimpses should then be allocated by the relative lengths of the 'four' walls' emergence points or the expected time to emergence from behind each of the 'walls' of defilade dwell lengths $2L_1$, L_2, L_2, $2L_3$ which total 1,800 yards. Then x_1 should get $2L_1/1800 = 0.556$ fraction of the glimpses, x_4 should get $2L_3/1800 = 0.389$ of the glimpses, and x_2 and x_3 each should get $L_2/1800 = 0.028$ fraction of the glimpses. And this should continue only for the first 40 seconds, after which time if the walker is not seen, the glimpses will be allocated between x_1 and x_4 in proportion to their remaining dwell lengths of $2L_1/1700 = 0.588$ and 2L3/1700 = 0.412 respectively.

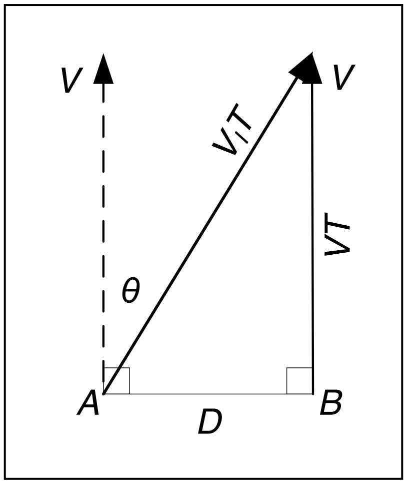
However, after 9.36 minutes the observer will abandon scanning x_4 and concentrate his entire attention on x_1 until 13.3 minutes, at which time scanning will be terminated with the report that 'She is not there.' (35%)

Problem 9: Rendezvous at Sea (10 points)

Two ships are steaming abeam, 50 miles apart, on parallel courses, and at the same speed of 15 knots. The ship A on the left is ordered to rendezvous with the ship B on the right as soon as possible. Ship B will maintain its course and speed. Ship A's maximum speed is 25 knots. A) What course and speed change will enable Ship A to comply with the minimum time to rendezvous requirement? B) What then is the course change and minimum time to rendezvous from the time that Ship A changes speed and course? (Hint: The speed of one knot equals one nautical mile per hour. 1 nautical mile = 1.15078 miles)

Solution: A) The student should draw a figure representing the situation that looks something like the one given here. The critical geometry is indicated by the solid line triangle. Solving the triangle gives

$$V_I^2 T^2 = D^2 + V^2 T^2 \rightarrow T^2 = \frac{D^2}{V_I^2 - V^2}$$



Minimizing T^2 will also minimize T. By inspection, we see that maximizing V_I will then minimize T. Therefore, Ship A will accelerate to its maximum speed for the duration T. The course change θ is obtained from the triangle as

$$\tan\theta = \frac{D}{VT} \rightarrow \theta = \tan^{-1}\frac{D}{VT}, \text{ or for}$$

$$T = \frac{D}{\sqrt{V_I^2 - V^2}},$$

$$\theta = \tan^{-1}\frac{D}{VT} \rightarrow \tan^{-1}\frac{\sqrt{V_I^2 - V^2}}{V}$$

Here we see that the course change for rendezvous does not depend on the separation distance between the ships. (75%)

B) To obtain the numerical solution using the above relations, we first convert the separation distance into nautical miles, giving $D = 50/1.15078 = 43.45$ nautical miles. Then using $V_I = 25$ knots, we get

$$T_{\min} = \frac{D}{\sqrt{V_I^2 - V^2}} = \frac{43.45}{\sqrt{25^2 - 15^2}} = 2.173 \text{ hours } = \text{ 2 hrs 10 min 23 sec.}$$

And Ship A's course change to starboard (right) is computed as

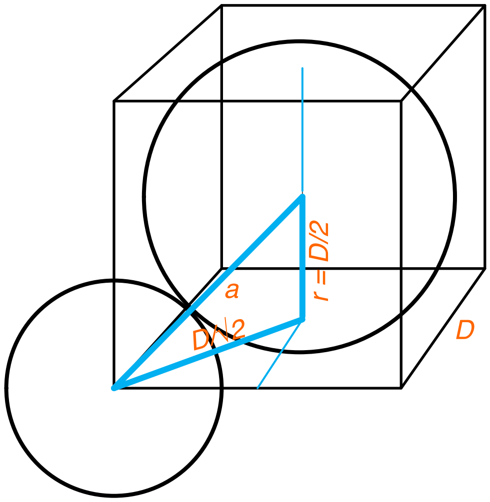
$$\theta = \tan^{-1}\frac{\sqrt{V_I^2 - V^2}}{V} = \tan^{-1}\frac{20}{15} = 53.13 \text{ degrees.}$$

(25%)

Problem 10: Packing Loss (10 points)

A large volume is divided into cubes with side length D. Into each cube is placed a sphere of maximum volume. Then between these spheres are placed smaller spheres of maximum volume. What fraction of the large volume is consumed by such placement of spheres? (This is known as the *packing density*. The residual fraction is known as *packing loss*.)

Solution: The student should draw a picture that looks something like the one nearby. Since a sphere's volume varies monotonically with its radius, each cube contains a large (circumscribed) sphere of radius $r_L = D/2$. The smaller spheres of radius r_S, centered on the vertices of the cubes, are then placed between the larger spheres. An important insight here is that each smaller sphere contributes 1/8 of its volume to every cube in which it resides. This means that the eight corners of each cube together contribute the volume of one small sphere, allowing the packing loss to be



computed simply as the difference between the volume of the cube and the volumes of the spheres divided by the volume of the cube. (30%)

From the figure, *a* is the length of the line connecting the centers of the large and small spheres.

$$a^2 = \left(\frac{D}{\sqrt{2}}\right)^2 + \left(\frac{D}{2}\right)^2 = D^2\left(\frac{1}{2}+\frac{1}{4}\right) \to a = \frac{D\sqrt{3}}{2},$$

$$r_S = a - r_L.$$

The volume of the cube is D^3 and of a sphere is $4\pi r^3/3$. Therefore the packing loss *PL* is a straightforward calculation from

$$\begin{aligned} PL &= \frac{1}{D^3}\left[D^3 - \frac{4\pi}{3}\left(r_L^3 + r_S^3\right)\right], \\ &= \frac{1}{D^3}\left[D^3 - \frac{4\pi D^3}{3}\left(\frac{1}{8} + \frac{\left(\sqrt{3}-1\right)^3}{8}\right)\right], \\ &= 1 - \frac{\pi}{6}\left(6\sqrt{3}-9\right) = 1 - \pi\left(\sqrt{3}-\frac{3}{2}\right) = 0.271 \end{aligned}$$

(70%)