TechTest2015

**Merit Scholarship Examination
in the Sciences and Mathematics
given on 11 April 2015, and**

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Solutions Key

Problem 1: Sharing Lunch Cost (5 points)

Abe, Betty, and Charlie decide to have lunch and share the cost equally. Abe buys the sandwiches for $*S*, and Betty pays $*D* for the drinks. Derive the general algorithm for when they settle up that computes who pays whom how much? (Hint: Substitute real amounts to check your solution.)

Solution: The total cost of lunch is $T = S + D$, so each must be $$T/3$ poorer when accounts are settled. The easiest way to settle accounts is to establish a 'pot' into which each pays or take from so that the pot winds up empty. The sums from each are given below such that a negative sign signifies paying in and a positive sign denotes taking out.

$$\begin{aligned}&\text{Abe pays } (S-T/3)\operatorname{sgn}(S-T/3);\\&\text{Betty pays } (D-T/3)\operatorname{sgn}(D-T/3);\\&\text{Charlie pays } (0-T/3)\operatorname{sgn}(0-T/3) = -T/3;\end{aligned}$$

so that

$$(S-T/3)\operatorname{sgn}(S-T/3)+(D-T/3)\operatorname{sgn}(D-T/3)-T/3=T$$

As check, say, $S = 6$ and $D = 4$ giving $T = 10$. Then Abe takes out +\$2.67, Betty takes out +\$0.67, and Charlie puts in -\$3.33 thereby leaving the pot empty.

Problem 2: Differentiation (15 points)

(A) Find *dy/dx* for $y = \dfrac{\cos x}{ax^2}$. (33%)

(B) Find *dy/dx* for $y = \dfrac{\tan x + ce^{2x}}{ax^3} + \dfrac{\cos x}{4x}$. (67%)

Hint: Express as function of sub-functions and then use the chain rule.

Solution: Part A

$$\begin{aligned}y &= \frac{\cos x}{ax^2} = f_0(f_1, f_2) = \frac{f_1}{f_2}\\ \frac{dy}{dx} &= \frac{\partial y}{\partial f_1}\frac{df_1}{dx} + \frac{\partial y}{\partial f_2}\frac{df_2}{dx} = \frac{1}{f_2}(-\sin x) + \frac{-f_1}{f_2^2}(2ax)\\ &= \frac{1}{ax^2}(-\sin x) + \frac{-\cos x}{a^2x^4}(2ax) = -\frac{1}{ax^2}\left(\sin x + \frac{2\cos x}{x}\right)\end{aligned}$$

(33%)

Part (B)

$$y=\frac{\tan x+ce^{2x}}{ax^3}+\frac{\cos x}{4x}=f_0\left(f_{1,1}\left(f_{2,1},f_{2,2},f_{2,3}\right),f_{1,2}\left(f_{2,4},f_{2,5}\right)\right),$$

$$y=f_{1,1}+f_{1,2}=\frac{f_{2,1}+f_{2,2}}{f_{2,3}}+\frac{f_{2,4}}{f_{2,5}},$$

$$\frac{dy}{dx}=\frac{\partial y}{\partial f_{1,1}}\frac{df_{1,1}}{dx}+\frac{\partial y}{\partial f_{1,2}}\frac{df_{1,2}}{dx}=1\frac{df_{1,1}}{dx}+1\frac{df_{1,2}}{dx},$$

$$\frac{dy}{dx}=\left(\frac{\partial f_{1,1}}{\partial f_{2,1}}\frac{df_{2,1}}{dx}+\frac{\partial f_{1,1}}{\partial f_{2,2}}\frac{df_{2,2}}{dx}+\frac{\partial f_{1,1}}{\partial f_{2,3}}\frac{df_{2,3}}{dx}\right)+\left(\frac{\partial f_{1,2}}{\partial f_{2,4}}\frac{df_{2,4}}{dx}+\frac{\partial f_{1,2}}{\partial f_{2,5}}\frac{df_{2,5}}{dx}\right)$$

$$=\left(\frac{1}{f_{2,3}}\frac{df_{2,1}}{dx}+\frac{1}{f_{2,3}}\frac{df_{2,2}}{dx}-\frac{f_{2,1}+f_{2,2}}{f_{2,3}^2}\frac{df_{2,3}}{dx}\right)+\left(\frac{1}{f_{2,5}}\frac{df_{2,4}}{dx}-\frac{f_{2,4}}{f_{2,5}^2}\frac{df_{2,5}}{dx}\right)$$

$$=\left(\frac{1}{f_{2,3}}\sec^2 x+\frac{1}{f_{2,3}}2ce^{2x}-\frac{f_{2,1}+f_{2,2}}{f_{2,3}^2}3ax^2\right)+\left(-\frac{1}{f_{2,5}}\sin x-\frac{f_{2,4}}{f_{2,5}^2}4\right)$$

$$=\left(\frac{1}{ax^3}\sec^2 x+\frac{1}{ax^3}2ce^{2x}-\frac{\tan x+ce^{2x}}{\left(ax^3\right)^2}3ax^2\right)+\left(-\frac{1}{4x}\sin x-\frac{\cos x}{(4x)^2}4\right)$$

$$=\frac{1}{ax^3}\left(\sec^2 x+2ce^{2x}-\frac{3\left(\tan x+ce^{2x}\right)}{x}\right)-\frac{1}{4x}\left(\sin x+\frac{\cos x}{x}\right)$$

(67%)

3: The Catch-up Game (10 points)

Say you are filling a storage tank of volume *V* with a liquid, and the job has to be done in time interval *T*. You found an adjustable rate pump with a stated maximum pumping rate *R* that you determine will do the job. You adjust the pump's rate to a setting so as to fill the tank before *T* has passed, and start pumping. After some time $T1 < T$ you return to check on progress and observe that $V1 < V$ has been pumped into the tank. From this you conclude that you had erroneously set the pumping rate too low. To complete the job as required you must now readjust the pumping rate.

What convinced you to select the pump you are using? How did you determine that the initial pump setting was too low? To what minimum pumping rate must you now set the pump so as to finish the job as prescribed? Finally, describe the performance requirement of your pump in order to determine whether it can do the job in the time remaining.

Solution: Given *V* and *T*, you ascertained that $R > V/T$ to select the pump. In time *T1* the tank should have been filled to at least $V1' = (T1/T)V$. Since $V1 < V1'$, you conclude that the pump rate setting was too low during *T1*. Now in the time remaining *T – T1* an additional volume *V – V1* must pumped into the tank. Therefore the pump must be set to a pumping rate of *R' ≥ (V-V1)/(T-T1).* To enable this, the pump's maximum pumping rate must satisfy *R ≥ R'*, else the pump will not serve to complete the job as required.

Problem 4: Range of Ancient Slingshot (10 points)

Derive the formula for how far from the center of rotation will a released weight of mass *M* impact the ground if it is spun at radius *R* with an angular rate *ω* given that its plane of rotation is a distance *H* above and parallel to the local horizon plane? (70%) What will that distance be if the rotational rate RPM = 240, *H* = 6 feet, *R* = 3 feet, and *M* = 4 ounces. (30%) (Hint: Ignore air friction but not gravity, and think of the kind of slingshot David used against Goliath. Read the problem statement carefully.)

Solution: Student should draw a labeled picture to indicate understanding of the problem's geometry. The path taken by *M* is resolved into two perpendicular components – horizontal and vertical. From the release point the horizontal distance is covered at constant velocity $v = \omega R$, and the vertical distance *H* is covered under the constant acceleration of gravity g. Using the well-known formula $s = s_0 + vT + \frac{1}{2}aT^2$ for distance *s* covered in time interval *T* from starting point s_0 by a point moving at a uniform velocity *v* and acceleration *a*, we calculate the time T for M to hit the ground from $H = \frac{1}{2}gT^2 \rightarrow T = \sqrt{\frac{2H}{g}}$. The horizontal distance *s* from the release point covered during *T* is then $s = vT = \omega RT$. Substituting gives

$$s = \omega R\sqrt{\frac{2H}{g}}, \quad s' = \sqrt{s^2 + R^2} = \sqrt{\frac{\omega^2 R^2 2H}{g} + R^2} = R\sqrt{\frac{2\omega^2 H}{g} + 1}$$

Where *s'* is the correct answer "from the center of rotation". (The student should note that *M* plays no role in the problem as stated.) (70%)

Doing the numerical example requires first converting RPM (revolutions per minute) into radians per second. One revolution = 2π radians, and one minute = 60 seconds. Therefore we convert by multiplying 120 RPM by two appropriate factors each equaling unity –

$$\frac{2\pi\text{*radians}}{1\text{*revolution}} = 1, \quad \frac{1\text{*minute}}{60\text{*seconds}} = 1,$$
$$\therefore \omega = 240\left(\frac{\text{revolutions}}{\text{minute}}\right)\left(\frac{2\pi\text{*radians}}{1\text{*revolution}}\right)\left(\frac{1\text{*minute}}{60\text{*seconds}}\right) = 8\pi\left(\frac{\text{radians}}{\text{second}}\right)$$

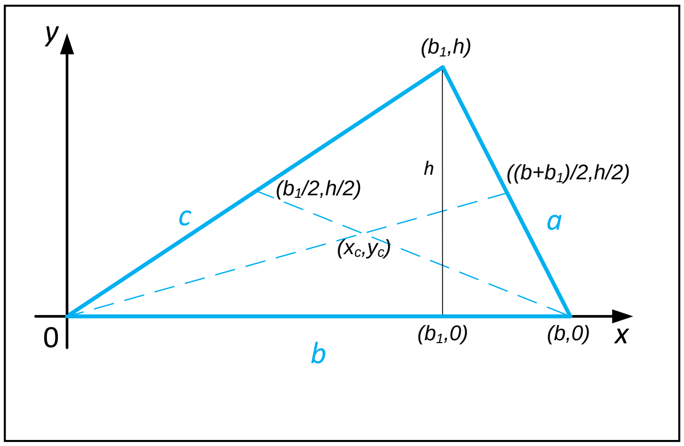
Then using g = 32 ft/sec^2 in the formula for *s'* lets us calculate the numerical answer as

$$s' = R\sqrt{\frac{2\omega^2 H}{g}+1} = 3\sqrt{\frac{2*(8\pi)^2*6}{32}+1} = 3\sqrt{\frac{2*(8\pi)^2*6}{32}+1} = 3\pi\sqrt{25} = 47.12 \text{ feet} \quad (30\%)$$

Problem 5: Centroid of a triangle (15 points)

Given a triangle with sides a,b,c such that $b > a + c$, place it into an appropriate Cartesian coordinate system and find the coordinates of its centroid. (80%) Find the coordinates of the triangle $(a,b,c) = (4,7,6)$. (Hint: recall that a triangle's centroid is at the common intersection of lines from each vertex to the middle of its opposite side.)

Solution: The student should draw a figure like shown, placing the long side on the x-axis. Then the derivation is straightforward, first calculating the triangle's altitude h from which can be written the mid-point coordinates of two sides. From these two lines are obtained which intersect at the centroid. The figure defines the needed terms for the derivation.



$$h^2 = c^2 - b_1^2 = a^2 - (b-b_1)^2$$

$$c^2 + b^2 - a^2 = 2bb_1 \rightarrow b_1 = \frac{c^2+b^2-a^2}{2b}$$

$$\therefore h = \sqrt{c^2 - \left(\frac{c^2+b^2-a^2}{2b}\right)^2}$$

The equations for the two blue dotted lines are respectively,

$$y_1 = \left(\frac{h}{b+b_1}\right)x,\quad y_2 = -\left(\frac{h/2}{b-b_1/2}\right)x + d \xrightarrow{y_2(b)=0} d = \left(\frac{bh}{2b-b_1}\right),\quad \therefore y_2 = \frac{bh}{2b-b_1}(-x+b)$$

At (x_C,y_C) we must have $y_1 = y_2$, therefore $\left(\frac{h}{b+b_1}\right)x_C = \frac{h}{2b-b_1}(-x_C+b)$, and solving for x_C gives

$$x_C = \frac{b+b_1}{3} = \frac{1}{3}\left(b + \frac{c^2+b^2-a^2}{2b}\right) = \left(\frac{c^2+3b^2-a^2}{6b}\right).$$

And substituting into $y_1(x)$ we get

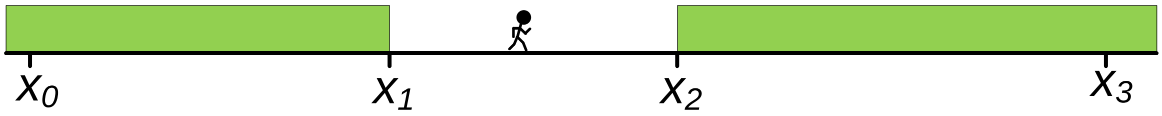
$$y_C = y_1(x_C) = \left(\frac{h}{b+b_1}\right)x_C = \left(\frac{h}{b+b_1}\right)\frac{b+b_1}{3} = \frac{h}{3},$$
$$= \frac{1}{3}\sqrt{c^2 - \left(\frac{c^2+b^2-a^2}{2b}\right)^2}.$$

(80%)

Finally, letting $(a,b,c) = (4,7,6)$ and substituting into the derived centroid coordinate formulas, we obtain the coordinates $x_C = 3.98$ and $y_C = 1.14$. (20%)

Problem 6: The Length of a Walking Path (20 points)

There is a park across the street from your office building. Sometime every morning in the park there is a woman who is known to walk back and forth on a fixed length path only a portion of which is visible to you (see figure). At a random time when you look into the space $[x_1,x_2]$ where you could see the constant speed walker, she is either visible or not. You want to estimate the total length $S_T = x_3 - x_0$ of the path on which she walks back and forth, but all you know is the length $S_V = x_2 - x_1$ of the visible part of her path. Suddenly it occurs to you how you can estimate S_T by just recording N_V, the number of times she is visible when you first look, and N_T, the total number of times you have looked over some period of weeks or months. Derive the formula for estimating S_T (70%), and demonstrate the sanity of your solution (30%). (Hint: given the problem parameters, what are the ways to express the probability that you see the walker when you first look for her? For the record, this problem was inspired by TT2014's Problem 8.)



Solution: The times that the walker is hidden, she may be behind $[x_0,x_1]$ or $[x_2,x_3]$ walking in either direction. The ratio N_V / N_T gives us the (frequentist) estimate of P. The probability P of seeing the walker at a randomly timed glimpse is also the fraction of time she would be visible heading in either direction on her back and forth path. To correctly calculate this fraction we need to 'fold out' the hidden lengths $S_{H1} = x_1 - x_0$ and $S_{H2} = x_3 - x_2$. And then consider that the walker can be anywhere with equal likelihood, walking in either direction, on this folded out virtual path of length $2S_{H1} + 2S_{H2} + S_V$. Since she walks at constant speed, the fraction of time she is visible is then equal to the fraction of visible part S_V to the virtual path length. This gives us P from which we can derive the estimating formula for S_T.

$$\frac{N_V}{N_T} = P = \frac{S_V}{2(S_{H1} + S_{H2}) + S_V},$$

$$S_{H1} + S_{H2} = \frac{(1-P)}{2P} S_V,$$

$$S_T = S_{H1} + S_{H2} + S_V = \left[1 + \frac{(1-P)}{2P}\right] S_V,$$

$$S_T = \left[\frac{1+P}{2P}\right] S_V$$

(70%)

The sanity check for the above S_T formula is argued by letting P approach its limits of zero and unity. P approaches zero when one seldom sees the walker, which means that S_T must be much larger than S_V (in the limit approaching infinity). And that is exactly what the derived formula shows as $\lim_{P \to 0} \left\{ S_T = \left[\frac{1+P}{2P}\right] S_V \right\} = \infty$. Conversely we know that if we almost always see the walker, then P is close to unity (certainty), indicating that the total path length S_T is essentially equal to the length of the visible part. Confirmed by $\lim_{P \to 1} \left\{ S_T = \left[\frac{1+P}{2P}\right] S_V \right\} = S_V$. (30%)

Problem 7: Two Equal Future Values (5 points)

The future value FV_k of a present value amount PV invested at interest rate I per period for k periods is given by $FV_k = PV(1+I)^k$. An investor has invested an amount A at interest rate I_m for m periods, and at some time later a second investor invested the same amount A at interest rate I_n for n periods, at the end of which both investors discover that their appreciated amounts are now equal. What is the relationship between the two interest rates (15%), and, given the problem parameters, what is the second investor's interest rate that yielded the equal amounts (85%)?

Solution: Clearly $n < m$, and the investors' equal amounts after m and n periods respectively are then

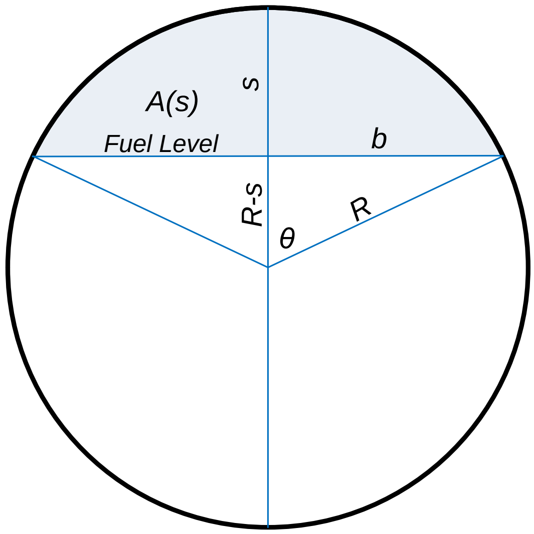
$$A(1+I_n)^n = A(1+I_m)^m \rightarrow (1+I_n)^n = (1+I_m)^m$$

$$(1+I_n)^1 = (1+I_m)^{m/n}$$

Since $m/n > 1$ we must have $1+I_n > 1+I_m \rightarrow I_n > I_m$ (15%). From the above we solve for I_n that gives the desired calculation.

$$I_n = (1+I_m)^{m/n} - 1 \ .$$

(85%)



Problem 8: How much heating oil remains (20 points)

A residential heating oil tank is in the shape of a right circular cylinder of known radius R and length L, and it is emplaced so that its cylindrical axis is horizontal. From the fill hole in the top of the tank the homeowner can measure the depth D of the fuel remaining. Since the total volume V_T of the tank is known, how is D converted into the volume of fuel remaining? (An extra 20% will be awarded for a sanity check of the derived formula.)

Solution: First and foremost, the student should draw an appropriately useful figure like shown that confirms his understanding of the problem, namely that we need only deal with a circular cross section and ignore the length of the tank. (20%)

From the figure we see that the measured depth is $D = 2R - s$, and $A(s)$ is the shaded area above the indicated fuel level. The volume is proportional to the cross sectional $A(2R) = \pi R^2$ so that the fuel remaining $V(D)$ can be computed from

$$V(s) = f(s)V_T = \left[1 - \frac{A(s)}{A(2R)}\right]V_T \ . \qquad (15\%)$$

Since $s = 2R - D$, our problem reduces to finding $A(s)$. From the figure the area $A(\theta)$ of the 2θ wide sector is

$$A(\theta) = \frac{2\theta}{2\pi}A(2R) = \theta R^2 = R^2 \cos^{-1}\left(\frac{R-s}{R}\right) \ .$$

To obtain $A(s)$ we will subtract the area A_T of the triangular part of the sector from $A(\theta)$. This requires that we first find b which is simply $b = \sqrt{R^2 - (R-s)^2} = \sqrt{2Rs - s^2}$. This gives us

$$A_T = b(R-s) = (R-s)\sqrt{2Rs - s^2}$$

so that

$$A(s) = A(\theta) - A_T = R^2 \cos^{-1}\left(\frac{R-s}{R}\right) - (R-s)\sqrt{2Rs - s^2}, \quad s \in [0, 2R] \ . \qquad (30\%)$$

Then substituting from above, we have the expression for the fuel remaining.

$$V(s) = \left[1 - \frac{A(s)}{A(2R)}\right] V_T$$

$$= \left[1 - \frac{R^2 \cos^{-1}\left(\frac{R-s}{R}\right) - (R-s)\sqrt{2Rs - s^2}}{\pi R^2}\right] V_T$$

And substituting $s = 2R - D$ gives the expression required in the problem statement.

$$V(D) == f(D) V_T = \left[1 - \frac{R^2 \cos^{-1}\left(\frac{D-R}{R}\right) - (D-R)\sqrt{D(2R-D)}}{\pi R^2}\right] V_T$$ (35%)

Sanity check: the student should substitute into the above formula D = 0, *R*, 2*R* to obtain values of 0, ½, 1 respectively for *f*(*D*). (20%)