Final Edit Copy

TechTest2016

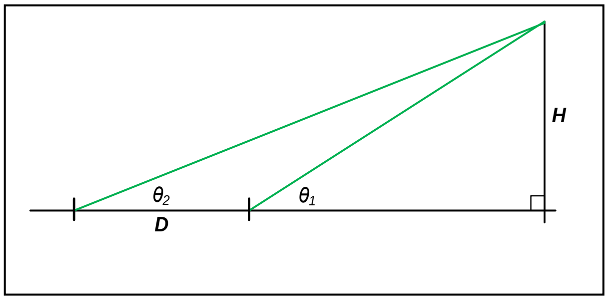
Merit Scholarship Examination
in the Sciences and Mathematics
given on 9 April 2016, and

sponsored by

The Sierra Economics and Science Foundation

Solutions Key

Problem 1: Height of an inaccessible cliff (5 points)



A survey team must find the relative height of a nearby mountain peak. They decide to take elevation angle measurements to the peak from two points on the flat and level valley floor located a known distance apart as shown in the figure. (A, 75%) Derive the height (*H*) of the mountain in terms of the three known quantities. (B, 25%) What is *H* when *D* = 2,000 ft, θ_1 = 45 deg, and θ_2 = 35 deg?

Solution: (A) Let *S* be the unknown distance to the peak's perpendicular that completes the definition of the right triangle. Then from the figure $\cot\theta_1 = \frac{S}{H}$, and $\cot\theta_2 = \frac{D+S}{H}$. Solving for and eliminating *S* from these two equations yields (accept either form of *H*)

$$H\cot\theta_1 = H\cot\theta_2 - D$$
$$H = \frac{-D}{\cot\theta_1 - \cot\theta_2}$$ (75%)

(B)

$$\frac{-2000}{\cot 45 - \cot 35} = \frac{-2000}{1.000 - 1.4281} = 4{,}671.3\text{ft} \ .$$ (25%)

Problem 2: Differentiation (15 points)

(A) Find *dy/dx* for $y = \frac{\cos x}{ax^2}$. (25%)

(B) Find *dy/dx* for $y = \frac{\tan x + ce^{2x}}{ax^3} + \frac{\cos x}{4x}$. (50%)

(C) Find *dy/dx* for $y = \frac{x^2\left(x+a\right)^3}{\left(bx+c\right)^4\left(hx+k\right)^5}$. (25%)

Hint: Express as function of sub-functions and then use the chain rule and/or the well-known product rule for $y = y_1y_2...y_N$, where

$$\frac{y'}{y} = \frac{dy/dx}{y} = \frac{y_1'}{y_1}\frac{y_2'}{y_2}\cdots\frac{y_N'}{y_N}$$

Solution: Part (A)

$$y=\frac{\cos x}{ax^2}=f_0\left(f_1,f_2\right)=\frac{f_1}{f_2}$$

$$\frac{dy}{dx}=\frac{\partial y}{\partial f_1}\frac{df_1}{dx}+\frac{\partial y}{\partial f_2}\frac{df_2}{dx}=\frac{1}{f_2}\left(-\sin x\right)+\frac{-f_1}{f_2^2}\left(2ax\right)$$

$$=\frac{1}{ax^2}\left(-\sin x\right)+\frac{-\cos x}{a^2x^4}\left(2ax\right)=-\frac{1}{ax^2}\left(\sin x+\frac{2\cos x}{x}\right)$$

(25%)

Part (B)

$$y=\frac{\tan x+ce^{2x}}{ax^3}+\frac{\cos x}{4x}=f_0\left[f_{1,1}\left(f_{2,1},f_{2,2},f_{2,3}\right),f_{1,2}\left(f_{2,4},f_{2,5}\right)\right],$$

$$y=f_{1,1}+f_{1,2}=\frac{f_{2,1}+f_{2,2}}{f_{2,3}}+\frac{f_{2,4}}{f_{2,5}},$$

$$\frac{dy}{dx}=\frac{\partial y}{\partial f_{1,1}}\frac{df_{1,1}}{dx}+\frac{\partial y}{\partial f_{1,2}}\frac{df_{1,2}}{dx}=1\frac{df_{1,1}}{dx}+1\frac{df_{1,2}}{dx},$$

$$\frac{dy}{dx}=\left(\frac{\partial f_{1,1}}{\partial f_{2,1}}\frac{df_{2,1}}{dx}+\frac{\partial f_{1,1}}{\partial f_{2,2}}\frac{df_{2,2}}{dx}+\frac{\partial f_{1,1}}{\partial f_{2,3}}\frac{df_{2,3}}{dx}\right)+\left(\frac{\partial f_{1,2}}{\partial f_{2,4}}\frac{df_{2,4}}{dx}+\frac{\partial f_{1,2}}{\partial f_{2,5}}\frac{df_{2,5}}{dx}\right)$$

$$=\left(\frac{1}{f_{2,3}}\frac{df_{2,1}}{dx}+\frac{1}{f_{2,3}}\frac{df_{2,2}}{dx}-\frac{f_{2,1}+f_{2,2}}{f_{2,3}^2}\frac{df_{2,3}}{dx}\right)+\left(\frac{1}{f_{2,5}}\frac{df_{2,4}}{dx}-\frac{f_{2,4}}{f_{2,5}^2}\frac{df_{2,5}}{dx}\right)$$

$$=\left(\frac{1}{f_{2,3}}\sec^2x+\frac{1}{f_{2,3}}2ce^{2x}-\frac{f_{2,1}+f_{2,2}}{f_{2,3}^2}3ax^2\right)+\left(-\frac{1}{f_{2,5}}\sin x-\frac{f_{2,4}}{f_{2,5}^2}4\right)$$

$$=\left(\frac{1}{ax^3}\sec^2x+\frac{1}{ax^3}2ce^{2x}-\frac{\tan x+ce^{2x}}{\left(ax^3\right)^2}3ax^2\right)+\left(-\frac{1}{4x}\sin x-\frac{\cos x}{\left(4x\right)^2}4\right)$$

$$=\frac{1}{ax^3}\left(\sec^2x+2ce^{2x}-\frac{3\left(\tan x+ce^{2x}\right)}{x}\right)-\frac{1}{4x}\left(\sin x+\frac{\cos x}{x}\right)$$

(50%)

Part (C)

$$y=\frac{x^2\left(x+a\right)^3}{\left(bx+c\right)^4\left(hx+k\right)^5}=x^2\left(x+a\right)^3\left(bx+c\right)^{-4}\left(hx+k\right)^{-5}$$

$$y'=\frac{dy}{dx}=y\left[\frac{2x}{x^2}+\frac{3\left(x+a\right)^2}{\left(x+a\right)^3}-\frac{4b\left(bx+c\right)^4}{\left(bx+c\right)^5}-\frac{5h\left(hx+k\right)^5}{\left(hx+k\right)^6}\right]$$

$$y'=\left[\frac{x^2\left(x+a\right)^3}{\left(bx+c\right)^4\left(hx+k\right)^5}\right]\left[\frac{2}{x}+\frac{3}{\left(x+a\right)}-\frac{4b}{\left(bx+c\right)}-\frac{5h}{\left(hx+k\right)}\right]$$

(25%)

3: Drought Management (12 points)

You are managing the volume and water level of a dammed reservoir for which the maximum capacity surface area is a rectangle measuring one mile wide and two miles long. The width cross section of the reservoir is an isosceles triangle whose apex (depth) at maximum capacity is 500 feet underwater. Its current depth is 350 feet. You want the reservoir at 80% capacity before you start releasing water for downstream uses. If the inflow from the mountains is a constant 400 cubic feet per second, (A, 75%) what will be the maximum depth of the reservoir at 80% capacity? (B, 25%) How long will it take to reach that capacity?

Solution: (A, 75%) The reservoir's water volume at any depth can be viewed as a 'prism' whose capacity is its cross-sectional area times the length (2 miles). Writing this volume *V* as a function of depth (*h* ft) gives $V(h) = 10560\left[hW(h)/2\right] = 55756.8h^2$. Since for similar triangles the ratio $W(h)/h$ = 5280/500 is constant, we have $W(h) = 10.56h$ which yields the constant (in units of feet) in the above volume equation. This gives the maximum capacity of the reservoir as $V(500) = 55756.8*500^2 = 13.393\text{e}9\ \text{ft}^3$. 80% of this is $0.8* V(500) = 11.151\text{e}9\ \text{ft}^3$. At this volume the maximum depth of the reservoir will be

$$h = \sqrt{\frac{V(h)}{55756.8}} = \sqrt{\frac{11.151e9}{55756.8}} = 447.2 \text{ ft.}$$

(B, 25%) The current volume of the reservoir is $V(350) = 6.830\text{e}9\ \text{ft}^3$, leaving a volume of $0.8* V(500) - V(350) = 4.321\text{e}9\ \text{ft}^3$ left to fill. At the inflow rate this will take 4.321e9/400/3600/24 ≈ 125 days before water can be released for downstream uses.

Problem 4: Solve the Quadratic (5 points) - Derive the well-known solution for the quadratic equation $ax^2 + bx + c = 0$.

Solution: The standard solution involves the algebraic identity $x^2 + 2hx + h^2 = (x+h)^2$, usually labeled as 'completing the square' method. The original quadratic is first put into this form, and then the square root is taken of both sides that allows a straightforward solution for *x*, giving the two classical solutions obtained using the +/- square root values. (100%)

$$ax^2+bx+c=0\to x^2+\frac{b}{a}x+\frac{c}{a}=0\to x^2+\frac{b}{a}x=-\frac{c}{a}$$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2\xrightarrow[\text{Completed}]{\text{Square}}\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

$$x+\frac{b}{2a}=\pm\sqrt{-\frac{c}{a}+\left(\frac{b}{2a}\right)^2}\to x=-\frac{b}{2a}\pm\sqrt{-\frac{c}{a}+\left(\frac{b}{2a}\right)^2}$$

$$x=-\frac{b}{2a}\pm\frac{1}{2a}\sqrt{b^2-4ac}\to\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

Problem 5: Annual growth rates (10 points)

(A, 50%) Given that deposits are compounded quarterly (on the last day of the quarter), then for four quarterly growth rates r_1 through r_4, what will be the annualized percent growth for a certificate of deposit? (B, 50%) For continuous compounding an account with an initial deposit of A_0 another bank will pay $A(T) = A_0\, e^{rT}$ for deposits withdrawn after T days where r is the continuous per day appreciation rate. Then what is the equivalent annualized interest rate R for continuous compounding that the bank will pay over N years?

Solution: (A) The annualized growth is simply the result of successively applying (compounding) the quarterly rates r_1, r_2, r_3, r_4 to an initial deposit of A_0 and then computing its appreciated percent R at the end of the year.

$$\underbrace{\left(\underbrace{\left(\underbrace{\left(\underbrace{\left(A_0\left(1+r_1\right)\right)}_{At\ end\ Qtr\ 1}\left(1+r_2\right)\right)}_{At\ end\ Qtr\ 2}\left(1+r_3\right)\right)}_{At\ end\ Qtr\ 3}\left(1+r_4\right)\right)}_{At\ end\ Qtr\ 4}=A_0\prod_{i=1}^{4}\left(1+r_i\right)=A_0\left(1+R\right),$$

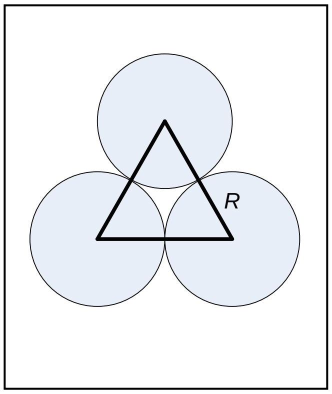
(50%)

then solve for $R=\prod_{i=1}^{4}\left(1+r_i\right)-1$

(B) Given 1 year = 365 days, and $N = T/365$ years, we can therefore argue as follows:
$A_0e^{rT}=A_0e^{RN}=A_0e^{R(T/365)}\xrightarrow{T=365}e^{r365}=e^{R(1)}$, giving $R = 365r$. (50%)

Problem 6: Packing Circles (5 points)

How would you place or 'pack' non-overlapping circles of radius R on a plane in such a way as to achieve maximum coverage or the largest packing density? What is the packing density ρ of your configuration? Hint: Packing density is the fraction of the plane's area covered by the circles.



Solution: The correct answer is hexagonal close packing whose repeating element that tiles the plane is the equilateral triangle shown in the figure. The student should draw such a figure to indicate her packing configuration. The packing density can then be written down by inspection as the area inside the equilateral triangle covered by circles - three 60 degree sectors or half a circle – divided by the area of the triangle. The density therefore is

$$\rho = \frac{\frac{\pi R^2}{2}}{\frac{(2R)\left(\sqrt{3}R\right)}{2}} = \frac{\pi}{2\sqrt{3}} = 0.9069 \ .$$

Problem 7: A Bit of Algebra (6 points)

(A, 70%) Solve $x^3 = bx + a^3 - ab$, for real constants a, b, and (B, 30%) specify conditions for all roots to be real.

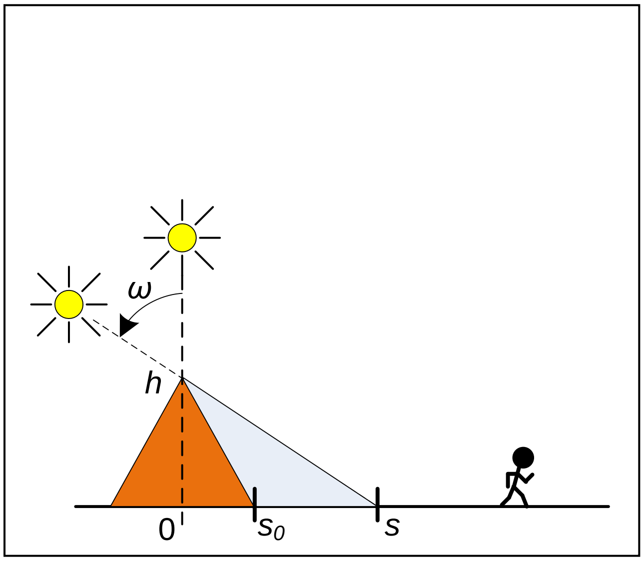
Solution: (A) Rearrange to get $x^3 - a^3 = b(x-a)$ which shows clearly that $x = a$ is real and one of the three roots of this polynomial. Then divide both sides by x-a, next divide out the fraction, and solve the resulting quadratic for the remaining two roots (use Problem 4).

$$\frac{x^3 - a^3}{x - a} = b$$
$$x^2 + ax + a^2 = b$$
$$x^2 + ax + \left(a^2 - b\right) = 0$$
$$\boxed{x = \frac{1}{2}\left[-a \pm \sqrt{a^2 - 4\left(a^2 - b\right)}\right] = \frac{1}{2}\left[-a \pm \sqrt{4b - 3a^2}\right]}$$

(70%)

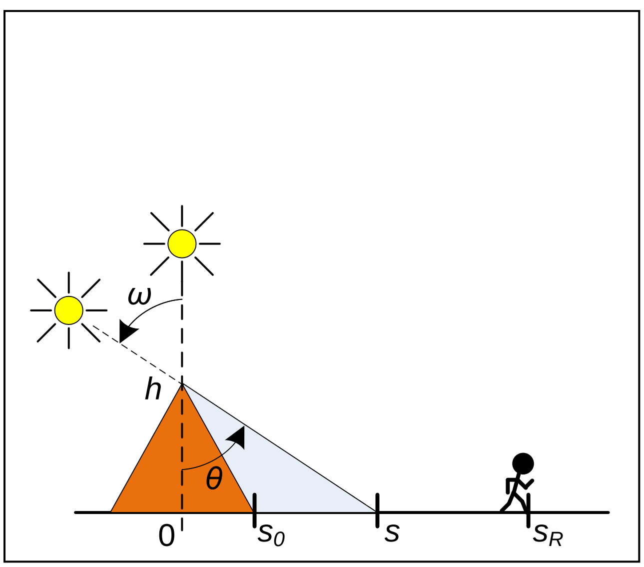
(B) For the above remaining two roots to be real the constants must satisfy $b \geq 3a^2/4$. (30%)

Problem 8: Transcendental Shadow Racing (30 points)



A runner decides to join the ranks of the famed Transcendental Shadow Racers and race the sun's shadow at the traditional east-west race course located at the foot of a high mountain on the equator. He assembles with other Shadow Racers on the plain at a point s_0 from the peak's vertical at the foot of a mountain of height h (see figure). At exactly noon (t = 0) the runners start running due east as the afternoon sun continues toward the west at an angular rate of ω. It starts casting the peak's shadow which, slowly at first, begins pursuing the racers. Our runner can run for hours at a speed of v_R. But alas, at $t = T$ the peak's shadow catches and passes our runner at $s(T)$.

(A, 60%) Derive the transcendental equation that must be solved to obtain T, and describe a graphical or iterative method for obtaining T ? (B, 30%) If h = 10,000 ft, s_0 = 2 miles, v_R = 5 mph, ω = 15 deg/hr, use your method from (A) to find an approximate solution for $s(T)$ to two decimal places. (C, 10%) At what speed did the shadow pass the runner at $s(T)$? Assume a flat earth. Hint: As you might guess, all the real action occurs in the last hour before sunset. Remember that at T the difference between the shadow's and runner's distances is zero.



Solution: (A, 60%) The student should draw the above figure and clearly label θ, the sun and shadow angle from noon, and also the runner's location s_R. S/he should then proceed with the derivation along the line indicated below.

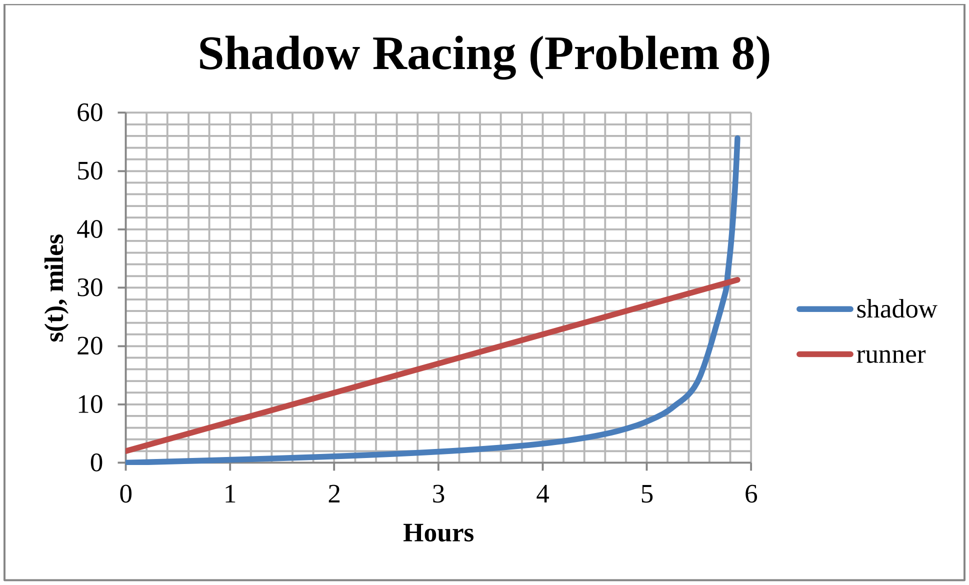
The sun's shadow at t is at distance

$$s(t) = h\tan\theta(t) = h\tan\omega t \ .$$

The runner's distance at t is

$$s_R(t) = s_0 + v_R t \ .$$

At $t = T$ we have $s(T) = s_R(T)$, giving the desired transcendental equation $h\tan\omega T = s_0 + v_R T$. This may be solved directly from the hint by computing the difference $s_R(T)$ - $s(T)$ starting at some $T \leq$ 5hrs. Then keep increasing T until the difference becomes negative which brackets the correct value of T. By halving the interval the correct answer is quickly obtained.



The student may also describe how the answer may be obtained from a plot of the two distances by then extracting T from their intersection as shown above.

Finally, an alternative and easily programmable iterative method is obtained by solving the transcendental equation for T and updating that value from the following relationship until it stabilizes. This occurs in about five iterations. (See also TT2012 Problem 3)

$$T_{i+1} = \frac{1}{\omega}\tan^{-1}\left(\frac{s_0 + v_R T_i}{h}\right).$$

(B, 30%) Given h = 10,000 ft, s_0 = 2 miles, v_R = 5 mph, ω = 15 deg/hr, using the most direct method to obtain T is to first recognize that the last hour before sunset starts at $t = T = 5$ hours. Starting at that value the student executes the procedure s/he described in Part A, and reduces the difference until $s_R(T) = s(T)$ = 30.825 miles at T = 5.765 = 5.77 hours. Also give credit if the student attempts to obtain the answer graphically to at least one decimal place by constructing any relevant part of the above figure.

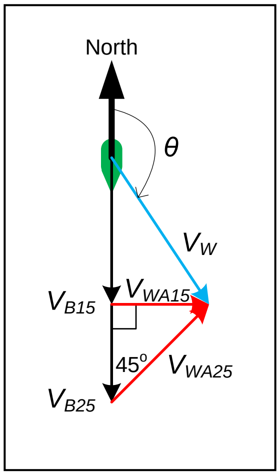
(C, 10%) The speed of the peak's shadow is $\dot{s}(t) = \frac{ds(t)}{dt} = \frac{d}{dt}(h\tan\omega t) = h\omega\sec^2\omega t$. Evaluating $\dot{s}(T)$ yields the desired answer.

$$\dot{s}(T) = h\omega\sec^2\omega T$$

$$= \frac{10000}{5280}\frac{15}{180/\pi}\sec^2\frac{15\bullet 5.77}{180/\pi} = 0.49583\sec^2 1.51 = 136.9\text{mph}$$

Problem 9: Velocity Vectors Problem (12 points)

A motorboat is travelling due south at 15mph. Its pilot notices that the apparent wind is blowing exactly from the west. He increases the boat's speed to 25mph and the apparent wind direction is now exactly from the southwest. What is the wind's real constant velocity V_W – i.e. its real direction (measured from true north) and speed? (A, 75%) first Derive the general case model for V_W, and then (B, 25%) calculate its numerical values. Hint: Recall that a vector is a two-dimensional quantity having both magnitude and direction. Draw and label the vector diagram of all the real and apparent velocities and solve.



Solution: The student should draw the shown vector diagram indicating the boat's speeds and the relevant 90 and 45 degree angles that result from the fact that the boat's velocity and the apparent wind velocity must always sum to the true wind velocity. This allows calculation of the simple triangles involved that yield the required true wind vector V_W.

For clarity let the magnitude (speed) of each subscripted velocity vector V_i be given by S_i. We then proceed as follows.

(A) From the geometry $S_{WA15} = S_{B25} - S_{B15}$, which gives

$$S_W = \sqrt{S_{WA15}^2 + S_{B15}^2} = \sqrt{\left(S_{B25} - S_{B15}\right)^2 + S_{B15}^2}$$

$$\tan\left(180 - \theta\right) = \frac{S_{B25} - S_{B15}}{S_{B15}} \longrightarrow \theta = 180 - \tan^{-1}\left(\frac{S_{B25} - S_{B15}}{S_{B15}}\right)$$

(75%)

(B) Substituting numerical values gives

$$S_W = \sqrt{\left(25 - 15\right)^2 + 15^2} = 18\text{mph}$$

$$\theta = 180 - \tan^{-1}\left(\frac{25 - 15}{15}\right) = 146.3\text{deg}$$

(25%)