Final Edit Copy

TechTest2017

Merit Scholarship Examination
in the Sciences and Mathematics
given on 1 April 2017, and

sponsored by

The Sierra Economics and Science Foundation

Solutions Key

Problem 1: Groove spacing on a vinyl record (5 points) A vinyl record rotates at 33 1/3 RPM and plays for 22.5 minutes on one side. The record's grooves are contained between the radii of 15cm and 6cm. What is the spacing between the grooves?

Solution: The easiest way to solve this is to realize that the groove spacing, g, times the total number of times, N_R, the record rotates should equal the difference of the radii ($R_2 - R_1$) where the grooves are located. Therefore

$$N_R = RPM * \text{minutes} = 33.33... * 22.5 = 750$$

$$N_R g = R_2 - R_1 \rightarrow g = \frac{R_2 - R_1}{N_R} = \frac{15-6}{750} = 0.012 \text{ cm}$$

Problem 2: Differentiation (10 points)

(A) Find dy/dx for $y = \dfrac{\cos x}{ax^2}$. (25%)

(B) Find dy/dx for $y = \dfrac{\tan x + ce^{2x}}{ax^3} + \dfrac{\cos x}{4x}$. (50%)

(C) Find dy/dx for $y = \dfrac{x^2 (x+a)^3}{(bx+c)^4 (hx+k)^5}$. (25%)

Hint: Express as function of sub-functions and then use the chain rule and/or the well-known product rule for $y = y_1 y_2 ... y_N$, where

$$\frac{y'}{y} = \frac{dy/dx}{y} = \frac{y_1'}{y_1} + \frac{y_2'}{y_2} + ... + \frac{y_N'}{y_N}$$

Solution: Part (A)

$$y = \frac{\cos x}{ax^2} = f_0(f_1, f_2) = \frac{f_1}{f_2}$$

$$\frac{dy}{dx} = \frac{\partial y}{\partial f_1}\frac{df_1}{dx} + \frac{\partial y}{\partial f_2}\frac{df_2}{dx} = \frac{1}{f_2}(-\sin x) + \frac{-f_1}{f_2^2}(2ax)$$

$$= \frac{1}{ax^2}(-\sin x) + \frac{-\cos x}{a^2x^4}(2ax) = -\frac{1}{ax^2}\left(\sin x + \frac{2\cos x}{x}\right)$$

(25%)

Part (B)

$$y=\frac{\tan x+ce^{2x}}{ax^3}+\frac{\cos x}{4x}=f_0\left[f_{1,1}\left(f_{2,1},f_{2,2},f_{2,3}\right),f_{1,2}\left(f_{2,4},f_{2,5}\right)\right],$$

$$y=f_{1,1}+f_{1,2}=\frac{f_{2,1}+f_{2,2}}{f_{2,3}}+\frac{f_{2,4}}{f_{2,5}},$$

$$\frac{dy}{dx}=\frac{\partial y}{\partial f_{1,1}}\frac{df_{1,1}}{dx}+\frac{\partial y}{\partial f_{1,2}}\frac{df_{1,2}}{dx}=1\frac{df_{1,1}}{dx}+1\frac{df_{1,2}}{dx},$$

$$\frac{dy}{dx}=\left(\frac{\partial f_{1,1}}{\partial f_{2,1}}\frac{df_{2,1}}{dx}+\frac{\partial f_{1,1}}{\partial f_{2,2}}\frac{df_{2,2}}{dx}+\frac{\partial f_{1,1}}{\partial f_{2,3}}\frac{df_{2,3}}{dx}\right)+\left(\frac{\partial f_{1,2}}{\partial f_{2,4}}\frac{df_{2,4}}{dx}+\frac{\partial f_{1,2}}{\partial f_{2,5}}\frac{df_{2,5}}{dx}\right)$$

$$=\left(\frac{1}{f_{2,3}}\frac{df_{2,1}}{dx}+\frac{1}{f_{2,3}}\frac{df_{2,2}}{dx}-\frac{f_{2,1}+f_{2,2}}{f_{2,3}^2}\frac{df_{2,3}}{dx}\right)+\left(\frac{1}{f_{2,5}}\frac{df_{2,4}}{dx}-\frac{f_{2,4}}{f_{2,5}^2}\frac{df_{2,5}}{dx}\right)$$

$$=\left(\frac{1}{f_{2,3}}\sec^2 x+\frac{1}{f_{2,3}}2ce^{2x}-\frac{f_{2,1}+f_{2,2}}{f_{2,3}^2}3ax^2\right)+\left(-\frac{1}{f_{2,5}}\sin x-\frac{f_{2,4}}{f_{2,5}^2}4\right)$$

$$=\left(\frac{1}{ax^3}\sec^2 x+\frac{1}{ax^3}2ce^{2x}-\frac{\tan x+ce^{2x}}{\left(ax^3\right)^2}3ax^2\right)+\left(-\frac{1}{4x}\sin x-\frac{\cos x}{(4x)^2}4\right)$$

$$=\frac{1}{ax^3}\left(\sec^2 x+2ce^{2x}-\frac{3\left(\tan x+ce^{2x}\right)}{x}\right)-\frac{1}{4x}\left(\sin x+\frac{\cos x}{x}\right)$$ (50%)

Part (C)

$$y=\frac{x^2\left(x+a\right)^3}{\left(bx+c\right)^4\left(hx+k\right)^5}=x^2\left(x+a\right)^3\left(bx+c\right)^{-4}\left(hx+k\right)^{-5}$$

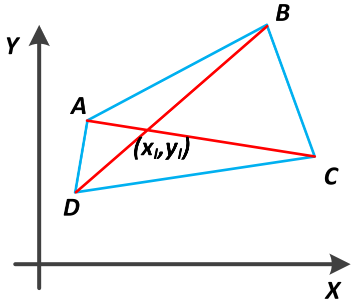
$$y'=\frac{dy}{dx}=y\left[\frac{2x}{x^2}+\frac{3\left(x+a\right)^2}{\left(x+a\right)^3}-\frac{4b\left(bx+c\right)^4}{\left(bx+c\right)^5}-\frac{5h\left(hx+k\right)^5}{\left(hx+k\right)^6}\right]$$ (25%)

$$y'=\left[\frac{x^2\left(x+a\right)^3}{\left(bx+c\right)^4\left(hx+k\right)^5}\right]\left[\frac{2}{x}+\frac{3}{\left(x+a\right)}-\frac{4b}{\left(bx+c\right)}-\frac{5h}{\left(hx+k\right)}\right]$$

Problem 3: Intersection in a Quadrilateral (10 points)

Given four non-colinear points A, B, C, D in a plane that form the vertices of a quadrilateral. Let their coordinates be (x_A, y_A), (x_B, y_{AB}), … . Let lines AC and BD connect the opposite vertices. (A, 80%) Derive the formulas for calculating the coordinates (x_I, y_I) of the intersection of these lines in terms of the coordinates of the vertices. Hint: To save work you can indicate your solution in terms of intermediate formulas for, say, slopes and intercepts. (B, 20%) What is the intersection for vertices (0, 7.5), (5.2, 12), (8.3, 1), (-1, -2)?

Solution: (A, 80%) Student should draw and label the quadrilateral to communicate correct term labels in the derivation. Referring to figure, it's easiest to proceed by expressing the two intersecting lines in their slope-intercept form. Then begin by writing the slope and intercept formulas for the intersecting lines.



$$y = m_{AC}x + b_{AC}, \quad y = m_{BD}x + b_{BD}$$
$$m_{AC} = \frac{y_A - y_C}{x_A - x_C}, \quad m_{BD} = \frac{y_B - y_D}{x_B - x_D}$$
$$b_{AC} = y_A - m_{AC}x_A, \quad b_{BD} = y_B - m_{BD}x_B$$

This lets us write the equations for the two intersecting lines at their intersection and then solve for (x_I, y_I) by first subtracting the equations to yield x_I, and then solve for y_I by substitution.

$$y_I = m_{AC}x_I + b_{AC}, \quad y_I = m_{BD}x_I + b_{BD}$$
$$0 = (m_{AC} - m_{BD})x_I + (b_{AC} - b_{BD}) \rightarrow x_I = \frac{(b_{BD} - b_{AC})}{(m_{AC} - m_{BD})}$$
$$y_I = m_{AC}\frac{(b_{BD} - b_{AC})}{(m_{AC} - m_{BD})} + b_{AC} = m_{BD}\frac{(b_{BD} - b_{AC})}{(m_{AC} - m_{BD})} + b_{BD}$$

(B, 20%) Substituting the numerical values for the vertices into the above gives (x_I, y_I) = (2.38, 5.64).

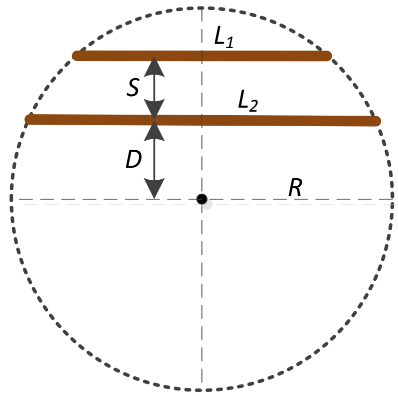
Problem 4: Archeological Dig (10 points) – On the excavation of the ruins of an ancient circular temple the archeologists know that there is a desired artifact buried at the center of the circle. However, the only remaining pieces of the temple are two parallel masonry walls of lengths $L_1 \leq L_2$ and a distance S apart that form the chords of the no longer visible circle. We know that the temple's center lies on the perpendicular bisector of the chords at some distance D from, say, the longer chord. Derive the formula for computing D. Hint: Check your result by letting the input parameters assume values for which you can easily deduce the answer.

Solution: Again the student should draw an appropriate picture of the situation labeling the parts to support the derivation and communicate its result. The simplest approach for obtaining *D* is to express and solve the two right triangles with *R* as their hypotenuse, and half of each chord as one of the perpendicular sides. Then

$$R^2 = \left(\frac{L_1}{2}\right)^2 + (S+D)^2, \quad R^2 = \left(\frac{L_2}{2}\right)^2 + D^2$$

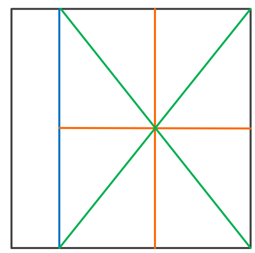
$$D^2 + 2SD + S^2 = R^2 - \left(\frac{L_1}{2}\right)^2, \quad D^2 = R^2 - \left(\frac{L_2}{2}\right)^2$$

$$2SD + S^2 = \left(\frac{L_2}{2}\right)^2 - \left(\frac{L_1}{2}\right)^2 \rightarrow D = \frac{L_2^2 - L_1^2}{8S} - \frac{S}{2}$$



Problem 5: Dividing the Square (5 points) – Divide a square into five equal area pieces with a minimum number of straight cuts. Give the dimensions of each piece. Hint: this is a freebie.

Solution: The minimum number of straight cuts is three. First, one fifth area of the square is cut off from one side by the blue line as shown in the figure. Then the bigger piece is cut twice as shown by either the red or green lines. For a unit square the end piece is 0.2 x 1.0. Each of the remaining four pieces made by the red cuts is 0.4 x 0.5. The green lines cut the remaining piece into four equilateral triangles, two of which have a base of 1.0 and altitude 0.4, and two have a base of 0.8 with an altitude of 0.5.

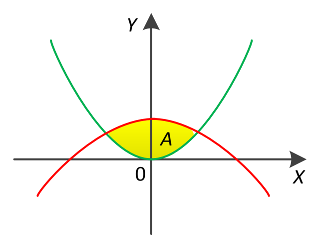


Problem 6: A Parabola-bounded Area (20 points)

(A, 85%) Derive the formula for the area *A* bounded by the two parabolas

$$y_1 = ax^2, \quad y_2 = bx^2 + c.$$

A configuration of the parabolas is shown in the nearby figure.
(B, 15%) Describe another configuration and check that your solution also works for that. Calculate *A* for constant values $a = 2$, $b = -1$, $c = 3$. Hint: Do a dimensional analysis to confirm that you have a sane solution.

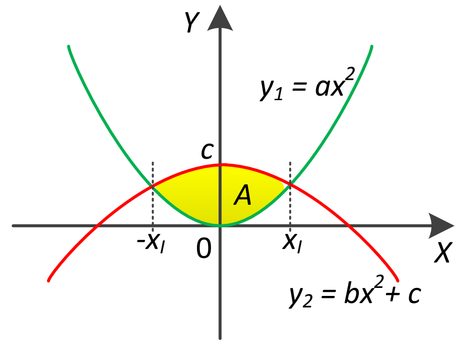


Solution: (A) The student should draw a figure similar to the one shown here to indicate his/her understanding of the problem set up. The desired area can be indicated as the difference of the areas under y_2 and y_1 computed between their intersections at x_I and $-x_I$, or alternatively just the area under the difference function $y_2 - y_1$. We begin by computing the intersections and proceed from there.

$$y_1(\pm x_I) = y_2(\pm x_I)$$

$$ax_I^2 = bx_I^2 + c \to x_I = \pm\sqrt{\frac{c}{a-b}}$$

$$A = \int_{-x_I}^{x_I} y_2 dx - \int_{-x_I}^{x_I} y_1 dx$$



We then perform the integrations and slog through the ensuing algebra.

$$A = \left(\frac{bx^3}{3} + cx\right)\Bigg|_{-x_I}^{x_I} - \frac{ax^3}{3}\Bigg|_{-x_I}^{x_I} = \left[\frac{(b-a)x^3}{3} + cx\right]\Bigg|_{-x_I}^{x_I}$$

$$= \left\{\frac{b-a}{3}\left(\frac{c}{a-b}\right)^{3/2} + c\left(\frac{c}{a-b}\right)^{1/2} - \frac{b-a}{3}\left[-\left(\frac{c}{a-b}\right)^{1/2}\right]^3 + c\left(\frac{c}{a-b}\right)^{1/2}\right\}$$

$$= \left\{2\left(\frac{b-a}{3}\right)\left(\frac{c}{a-b}\right)^{3/2} + 2c\left(\frac{c}{a-b}\right)^{1/2}\right\} = 2\left\{\left(\frac{b-a}{3}\right)\left(\frac{c}{a-b}\right)\left(\frac{c}{a-b}\right)^{1/2} + c\left(\frac{c}{a-b}\right)^{1/2}\right\}$$

$$= 2c\left(\frac{c}{a-b}\right)^{1/2}\underbrace{\left[\frac{b-a}{3(a-b)} + 1\right]}_{=2/3} \Rightarrow \boxed{A = \frac{4c}{3}\sqrt{\frac{c}{a-b}}} \text{ for } \frac{c}{a-b} \geq 0.$$

(80%)

(B, 20%) Understanding the behavior of parametized functions is a critical STEM skill. This simple problem exercises and measures early acquisition of this skill. The student should point out that the argument in the radical can never be negative for real areas, and then go on to observe that if $c < 0$ we must also have $a < b$ for a non-zero area. Important also to explain is that positive a,b values denote parabolas opening in the $+y$ direction and vice versa. And finally if $c > 0$ that $a > b$ meaning that a 'nested' y_2 cannot grow faster than y_1, else there will be no intersection of the functions and no bounded area. For the given constant values the answer is $A = 4$ which can be determined by inspection.

Problem 7: A Cards Probability (5 points)

What is the probability of drawing three-of-a-kind (e.g. 3 Kings, 3 tens, 3 deuces, …) with three randomly drawn cards from a regular deck of 52 cards?

Solution: There are two ways to get the correct answer of *P*(three of kind | three cards) = 6/(51*50) = 0.0024. The easiest is to consider that the first card drawn can be anything, but the probability of getting the same kind for the second card is 3/51. And having been successful with the first two, the probability of getting the correct third card is 2/50. The joint probability of these two related events is the product of their probabilities (3*2)/(51*50). The second method is to compute the probability as the number of ways to get three-of-a-kind from a 52 card deck and divide that by the total number of possible three card draws from the same deck. That gives

$$P = 13\binom{4}{3}\Big/\binom{52}{3} = \frac{13\left(\frac{4!}{3!1!}\right)}{\left(\frac{52!}{49!3!}\right)} = \frac{52}{\left(\frac{52\cdot 51\cdot 50}{6}\right)} = \frac{6}{51\cdot 50}$$

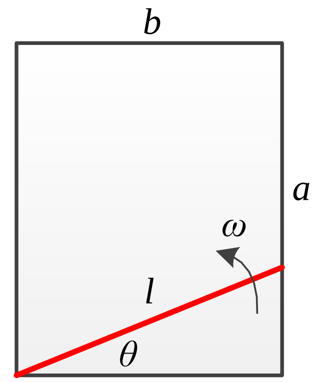
Problem 8: Currency Exchange Rate (5 points)

(A, 60%) Today the Euro/Dollar ratio is *X*, and it increased *Y*% from yesterday. What was that ratio yesterday, and what was the value of the dollar then (in euros)? (B, 40%) Today €0.90 = $1.00 which represents an increase of 1.5% from yesterday. What was yesterday's *X*?

Solution: (A, 60%) Today €/$ = *X* = (1 + *Y*)*Z* where *Z* was yesterday's ratio. Therefore *Z* = *X*/(1+*Y*). (B, 40%) From €0.90 = $1.00 we have *X* = €/$ = 1.00/0.90 = 1.11 or equivalently €1.00 = $1.11. Therefore *Z* = €/$ = 1.11/1.015 = 1.09 or equivalently €1.00 = $1.09.

Problem 9: Radiant in a rectangle (30 points)

From the corner of a rectangle with sides *a* > *b* we draw a radiant that makes the angle θ with its shorter adjacent side as shown in the figure. The radiant starts at $\theta = 0$ and sweeps CCW at a constant angular rate ω until it meets its longer adjacent side when $\theta = \pi/2$. (A, 40%) Derive the formulas for the length *l* of the radiant and its derivative as a function of θ. (B, 60%) Looking at the figure, describe the behavior of *l* as *a* gets very large – i.e. $a \gg b$. Using your derived formulas, demonstrate that your description is correct. For extra credit discuss and illustrate any interesting anomalies encountered during the sweep of the radiant.



Solution: (A, 40%) From the figure the student should intuit that *l* increases until it reaches the opposite corner and then decreases until reaching the longer adjacent side where $\theta = \pi/2$. The transition occurs when $\theta = \theta_{ab} = \tan^{-1}(a/b)$ which defines the boundary between the two dynamic regimes of $l(\theta)$. By inspection we write

$$\cos\theta = {}^{b}\!/_{l} \rightarrow l(\theta) = b\sec\theta, \quad \theta \in [0, \theta_{ab}]; \quad \sin\theta = {}^{a}\!/_{l} \rightarrow l(\theta) = a\csc\theta, \quad \theta \in \left(\theta_{ab}, {}^{\pi}\!/_{2}\right].$$

Noting that ${}^{d\theta}\!/_{dt} = \dot{\theta} = \omega$, we take the take the required derivatives using the chain rule as follows.

$$\dot{l}(\theta) = \frac{\partial l}{\partial \theta}\dot{\theta} = (b\tan\theta\sec\theta)\omega = b\omega\tan\theta\sec\theta, \text{ when } \theta \in [0, \theta_{ab}];$$
$$\dot{l}(\theta) = \frac{\partial l}{\partial \theta}\dot{\theta} = (-a\cot\theta\csc\theta)\omega = -a\omega\cot\theta\csc\theta, \text{ when } \theta \in \left(\theta_{ab}, {}^{\pi}\!/_{2}\right].$$

(B, 60%) Somewhere in these arguments the student should evince that the dynamic regimes depend on the ratio a/b since $a \gg b$ is equivalent to $a/b \gg 1$. The student should also intuit that as *a/b* gets very large, then $l(\theta_{ab}) \rightarrow a$, or $\lim_{a/b\to\infty} l(\theta_{ab}) = a$, and $\dot{l}(\theta_{ab}) \rightarrow$ large, or $\lim_{a/b\to\infty} \dot{l}(\theta_{ab}) = \infty$. In a similar manner for $\theta > \theta_{ab}$ we still have $l(\theta > \theta_{ab}) \simeq a$, or $\lim_{a/b\to\infty} l(\theta > \theta_{ab}) = a$, but now the rate of change of *l* approaches zero as seen from $\dot{l}(\theta > \theta_{ab}) \rightarrow$ small, or $\lim_{a/b\to\infty} \dot{l}(\theta > \theta_{ab}) = 0$. This should also be corroborated from the derived formulas for l and $\dot{l}$ in the two regimes by appropriately substituting from the figure and taking the above indicated limits.

$$\tan\theta = \frac{\sqrt{l^2 - b^2}}{b},\ \sec\theta = \frac{l}{b},\ \cot\theta = \frac{a}{\sqrt{l^2 - a^2}},\ \csc\theta = \frac{l}{a}.$$

(Extra Credit 15%) If the student includes a discussion of the 'jump' in $\dot{l}$ at θ_{ab}, especially pointing out that approaching θ_{ab} in the CW direction vs from the indicated CCW direction, then award up to an added maximum of 15% of the problem's point allowance. In short the student should indicate that $\dot{l}$ at θ_{ab}^- vs θ_{ab}^+ generates different answers. The magnitude of the jump is the difference

$$\Delta\dot{l} = \pm\left[\dot{l}\left(\theta_{ab}^-\right) - \dot{l}\left(\theta_{ab}^+\right)\right].$$

Note that at θ_{ab} we have $l^2 = a^2 + b^2$, and that using this in the above formulas at the 'jump corner' yields

$$\tan\theta_{ab}^- = \frac{a}{b},\ \sec\theta_{ab}^- = \frac{\sqrt{a^2 + b^2}}{b},\ \cot\theta_{ab}^+ = \frac{a}{b},\ \csc\theta_{ab}^+ = \frac{\sqrt{a^2 + b^2}}{a}.$$

Calculating the radiant's length rates of change at these points gives us the desired magnitude of the jump.

$$\begin{aligned}
\dot{l}\left(\theta_{ab}^-\right) &= \left.\frac{\partial l}{\partial\theta}\right|_{\theta_{ab}^-}\dot{\theta} = b\omega\tan\theta_{ab}^-\sec\theta_{ab}^-,\ \text{ when } \theta_{ab}^- \to \theta_{ab}; \\
&= b\omega\left(\frac{a}{b}\right)\frac{\sqrt{a^2 + b^2}}{b} = \omega\left(\frac{a}{b}\right)\sqrt{a^2 + b^2} \\
\dot{l}\left(\theta_{ab}^+\right) &= \left.\frac{\partial l}{\partial\theta}\right|_{\theta_{ab}^+}\dot{\theta} = -a\omega\cot\theta_{ab}^+\csc\theta_{ab}^+,\ \text{ when } \theta_{ab}^+ \to \theta_{ab}; \\
&= -a\omega\left(\frac{b}{a}\right)\frac{\sqrt{a^2 + b^2}}{a} = -\omega\left(\frac{b}{a}\right)\sqrt{a^2 + b^2}
\end{aligned}$$

$$\Delta\dot{l} = \pm\left[\dot{l}\left(\theta_{ab}^-\right) - \dot{l}\left(\theta_{ab}^+\right)\right] = \pm\omega\sqrt{a^2 + b^2}\left(\frac{a}{b} + \frac{b}{a}\right) = \pm\frac{\omega\left(a^2 + b^2\right)^{3/2}}{ab}.$$

And the ratio of the rates is a function only of *a/b* as expected.

$$\frac{\dot{l}\left(\theta_{ab}^-\right)}{\dot{l}\left(\theta_{ab}^+\right)} = -\left(\frac{a}{b}\right)^2$$

