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TechTest2018

Merit Scholarship Examination
in the Sciences and Mathematics
given on 18 March 2018, and

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Solutions Key

Problem 1: Liquid fertilizer recipe (10 points) A farmer mixes up a tank of liquid fertilizer by diluting its original concentration with a known volume of water. To test it, he takes out a measured container of known volume, and finds that it is too strong. He discards the contents of the container, tops off the tank with water to its original volume, and mixes it thoroughly. Once more he fills the measured container, tests it, and finds that it is still too strong. After discarding the container's contents, he again repeats the previous step, and the resulting mixture passes the test. He wants to adjust his original mixing recipe, and suddenly realizes that now the fertilizer in the tank is exactly half its original strength. (A, 80%) What is the volume of the tank as a function of the container's volume? (B, 20%) If the volume of the measured container is 2 liters, what is the volume of the tank?

Solution: (A) The easiest way to solve this is to realize that each dilution step leaves a fraction $(V_T - V_C)/V_T$ of the original concentration in the tank, where V_C is the volume of the container and V_T is the volume of the tank. Since the dilution step is applied twice, the total fraction of the original concentration is the product of dilution fractions which we are told equals 0.5 of the original strength. We then solve –

$$\left(\frac{V_T - V_C}{V_T}\right)^2 = \frac{1}{2} \rightarrow \frac{V_T^2}{2} - 2V_C V_T + V_C^2 = 0$$

$$V_T = \frac{2V_C \pm \sqrt{(-2V_C)^2 - 4\left(\frac{1}{2}\right)V_C^2}}{1} = 2V_C + \sqrt{2}V_C = 3.414V_C$$

(80%)

And (B) = $3.414V_C$ = 6.828 liters. (20%)

Problem 2: Three Racers (10 points) Three racers A, B, and C will compete in pairs by running a distance of length L each at their own constant speeds V_A, V_B, and V_C. After the first two races it turns out that A beat B by distance $D_{A/B}$, and B beat C by distance $D_{B/C}$. (A, 80%) Then by what distance can we expect A to beat C, i.e. what is the predicted value of $D_{A/C}$? (B, 20%) Given L = 100m, $D_{A/B}$ = 5m, $D_{B/C}$ = 10m, what will be $D_{A/C}$?

Solution: (A) Runner A beats B and completes the race in T_A seconds, giving

$$V_A = \frac{L}{T_A}, \text{ and } V_B = \frac{L - D_{A/B}}{T_A} .$$

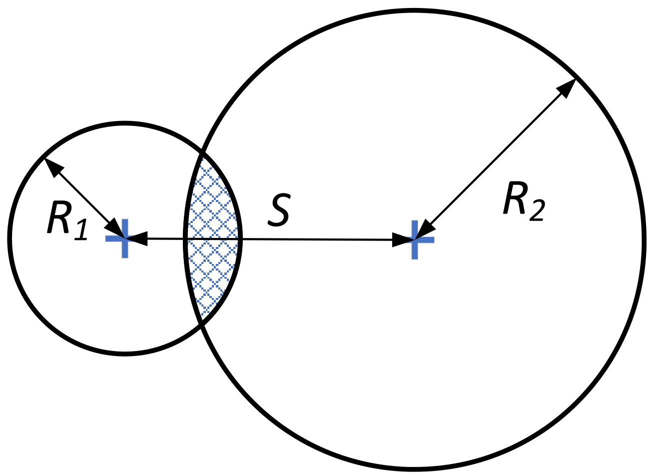
Runner B then beats C, and completes the race in T_B seconds, giving

$$T_B = \frac{L}{V_B} = \frac{LT_A}{L - D_{A/B}}, \text{ then } V_C = \frac{L - D_{B/C}}{T_B} = \frac{(L - D_{B/C})(L - D_{A/B})}{LT_A}.$$
$$\therefore D_{A/C} = L - V_C T_A = L - \frac{(L - D_{B/C})(L - D_{A/B})}{L}$$
(80%)

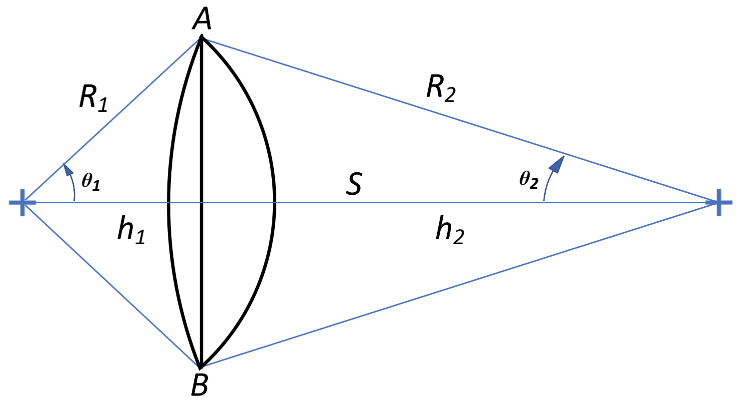
(B) Substituting into $D_{A/C}$ gives $D_{A/C} = 100 - \frac{90*95}{100} = 14.5\text{m}$ (20%)

Problem 3: Area of Intersecting Circles (25 points)

Given two circles having radii $R_1 \le R_2$ with centers separated by distance S intersect as shown. (A, 70%) Derive the formula(s?) for the intersecting (cross-hatched) area A_I as a function of the known R_1, R_2, and S. (B, 10%) Express and discuss the range and values of S at which A_I achieves its related extreme values. (C, 20%) Argue the correctness of your solutions to (A) over the range of S.



Solution: (A) The student should draw the relevant figure as shown here, labeling all the parts that will be germane in the derivation and to the derived formula.



The solution should then track generally in line with the following derivation. Alternate tracks are acceptable as long as they result in the correct final formula. The basic insight that launches the solution is initially recognizing that A_I is the sum of the two shown minor segments with a common chord $\left\|\overline{AB}\right\| = c$ within the intersection. The area of a minor segment is simply the area of its subtending sector minus the area of the triangle heights h_i formed by the radii and the segment's chord. The solution is best approached by introducing the indicated θ_i as intermediate variables.

$$S = h_1 + h_2$$

$$\text{Area of segment} = \frac{2\theta_i}{2\pi}\pi R_i^2 - \frac{1}{2}ch_i = \frac{2\theta_i}{2\pi}\pi R_i^2 - \frac{1}{2}(2R_i\sin\theta_i)(R_i\cos\theta_i)$$

$$= \frac{2\theta_i}{2}R_i^2 - \frac{1}{2}\left(R_i^2\sin 2\theta_i\right) = \frac{R_i^2}{2}\left[2\theta_i - \sin 2\theta_i\right]$$

Then the area of the intersection is $A_I = \frac{1}{2}\left[R_1^2\left(2\theta_1 - \sin 2\theta_1\right) + R_2^2\left(2\theta_2 - \sin 2\theta_2\right)\right]$. Now we need to express the θ_i as a function of the given parameters as required.

$$\left(\frac{c}{2}\right)^2 = R_2^2 - (S - h_1)^2 = R_1^2 - h_1^2 \rightarrow R_2^2 - S^2 + 2Sh_1 - h_1^2 = R_1^2 - h_1^2$$

$$h_1 = \frac{1}{2}\left(S - \frac{R_2^2 - R_1^2}{S}\right), \quad h_2 = S - h_1 = \frac{1}{2}\left(S + \frac{R_2^2 - R_1^2}{S}\right)$$

$$\cos\theta_1 = \frac{h_1}{R_1} = \frac{1}{2R_1}\left(S - \frac{R_2^2 - R_1^2}{S}\right), \quad \cos\theta_2 = \frac{h_2}{R_2} = \frac{1}{2R_2}\left(S + \frac{R_2^2 - R_1^2}{S}\right)$$

$$\theta_1 = \cos^{-1}\left\{\frac{1}{2R_1}\left(S - \frac{R_2^2 - R_1^2}{S}\right)\right\}, \quad \theta_2 = \cos^{-1}\left\{\frac{1}{2R_2}\left(S + \frac{R_2^2 - R_1^2}{S}\right)\right\}$$

Substituting these into the above equation for A_I gives us the final formula in terms of the given parameters.

$$A_I = \frac{R_1^2}{2}\left\{2\cos^{-1}\left[\frac{1}{2R_1}\left(S-\frac{R_2^2-R_1^2}{S}\right)\right]-\sin\left(2\cos^{-1}\left[\frac{1}{2R_1}\left(S-\frac{R_2^2-R_1^2}{S}\right)\right]\right)\right\}$$
$$+\frac{R_2^2}{2}\left\{2\cos^{-1}\left[\frac{1}{2R_2}\left(S+\frac{R_2^2-R_1^2}{S}\right)\right]-\sin\left(2\cos^{-1}\left[\frac{1}{2R_2}\left(S+\frac{R_2^2-R_1^2}{S}\right)\right]\right)\right\}$$

(70%)

(B) From the problem statement figure it is clear that when $S \geq R_1 + R_2$, then the intersection disappears and $A_I = 0$, therefore the maximum value is $S_{max} = R_1 + R_2$. Recalling that $R_1 \leq R_2$, then as S is made smaller the smaller circle is completely enclosed in the larger circle when $S = R_2 - R_1$ and A_I becomes the area of the smaller circle, remaining so even if S can be further diminished. Therefore $S_{min} = R_2 - R_1$ which, of course as expected, becomes zero when $R_1 = R_2$. (10%)

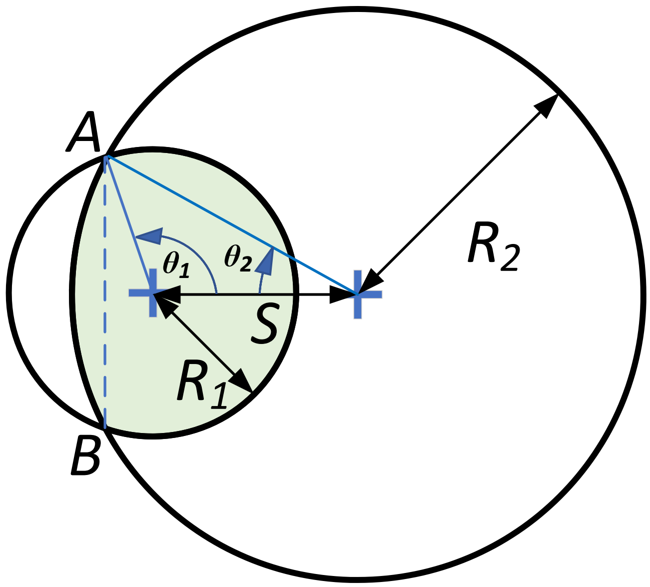
(C) The derived behavior of A_I as S decreases from S_{max} to S_{min} is most easily seen from its formulation in terms of the intersecting chord's subtending angles,

$$A_I = \frac{1}{2}\left[R_1^2\left(2\theta_1 - \sin 2\theta_1\right) + R_2^2\left(2\theta_2 - \sin 2\theta_2\right)\right] .$$

At S_{max} the circles osculate at a single point with $A_I = 0$ at which we have $\theta_1 = \theta_2 = 0$. The above equation gives the correct answer when these values are substituted. At the other extreme when the smaller circle is just completely enclosed within the bigger circle, they again osculate at a single point. But now we note that while θ_2 is again zero, θ_1 has rotated from zero to π. Substituting these values into the above equation yields the area of the smaller circle as desired.

$$A_I = \frac{1}{2}\left[R_1^2\left(2\pi - \sin 2\pi\right) + R_2^2\left(0 - \sin 0\right)\right] = \pi R_1^2$$

We now ask whether the above equation still gives the correct answer when $\theta_1 > \pi/2$ since that formulation was based on summing the areas of two minor segments, i.e. when segment's area was computed as the sector's area minus the related triangle's area. Now the area of the related triangle, that is outside the small circle's major sector and still bounded by the common chord, must be added to its major sector's area to obtain the area of its major segment. Note that the larger circle's contribution remains as the area of its minor segment, therefore the second term in the square brackets is unchanged. From the following figure we compute A_I as θ_1 increases from $\pi/2$ to π.



$$A_I = \underbrace{\underbrace{\frac{2\theta_1}{2\pi}\pi R_1^2}_{\text{Area of Sector}} + \underbrace{2\frac{1}{2}\left[R_1\sin(\pi-\theta_1)R_1\cos(\pi-\theta_1)\right]}_{\text{Area of 'external' triangle}}}_{\text{Area of Major Segment}} + \underbrace{\frac{R_2^2}{2}\left(2\theta_2-\sin 2\theta_2\right)}_{\text{Area of Minor Segment}}$$

$$= \frac{2\theta_1}{2}R_1^2 - \frac{1}{2}\left[R_1 2\sin(\theta_1)R_1\cos(\theta_1)\right] + \frac{R_2^2}{2}\left(2\theta_2-\sin 2\theta_2\right) \quad ,$$

$$= \frac{2\theta_1}{2}R_1^2 - \frac{1}{2}\left[R_1^2\sin(2\theta_1)\right] + \frac{R_2^2}{2}\left(2\theta_2-\sin 2\theta_2\right)$$

$$= \frac{1}{2}\left[R_1^2\left(2\theta_1-\sin 2\theta_1\right) + R_2^2\left(2\theta_2-\sin 2\theta_2\right)\right]$$

The initial derivation, based on arguments relating to A_I being the sum of minor segments, also holds when one of the segments changes to a major segment as θ_1 exceeds π /2. This confirms the generality of our initial derivation. (20%)

Problem 4: Growing cultures (10 points) Three tanks were seeded with equal starting weights w_0 of cultures A, B, C which were allowed to grow for a specified period of time. Each culture grew exponentially at its own rate – r_A, r_B, r_C. At the end of the time period T the rates and weights of the cultures were calculated and measured. However, an accident in the lab destroyed most of the recorded data with only T, the starting amount w_0, r_A, and two weight ratios or fractions – $f_{AB} = w_A/w_B$ and $f_{BC} = w_B/w_C$ – surviving. One of the experimenters examined the surviving data, and from that was able to recover the rest of the experiment's data. Show how all the weights, fractions, and rates were recovered.

Solution: From the weight ratio definitions we can derive the missing weight ratio –

$$w_A = f_{AB} w_B, \quad w_B = f_{BC} w_C \rightarrow w_C = \frac{w_B}{f_{BC}}$$

$$f_{AC} = \frac{w_A}{w_C} = \frac{f_{AB} w_B}{w_B / f_{BC}} = f_{AB} f_{BC}$$

(30%)

Letting T be the growth time period, the missing weights are recovered from –

$$w_A = w_0 e^{r_A T} \rightarrow \therefore w_B = \frac{w_A}{f_{AB}} = \frac{w_0 e^{r_A T}}{f_{AB}} = w_0 e^{r_B T}, \text{ and}$$

$$w_C = \frac{w_A}{f_{AC}} = \frac{w_0 e^{r_A T}}{f_{AB} f_{AC}} = w_0 e^{r_B T}$$

(35%)

Finally, solving for the missing growth rates, we have –

$$r_B = \frac{1}{T} \ln\left(\frac{w_B}{w_0}\right) = \frac{1}{T} \ln\left(\frac{e^{r_A T}}{f_{AB}}\right), \quad r_C = \frac{1}{T} \ln\left(\frac{w_C}{w_0}\right) = \frac{1}{T} \ln\left(\frac{e^{r_A T}}{f_{AB} f_{AC}}\right)$$

(35%)

Problem 5: Earth's Solar Energy Input (5 points)

Solar energy is becoming a more important component of human energy consumption. The total power output of the sun is 3.848e26 W (watts). (A, 50%) What is the maximum total solar power that impacts the earth if we ignore atmospheric effects? (B, 50%) If total annual human consumption of energy is 110e17 Wh (watt-hours), then what fraction of the sun's intercepted power output is the total power consumed by humans? Use 3,959 miles as the earth's radius, and 91,700,000 miles as the distance to the sun, and show your work. Assume impinging sun's rays are parallel. Hint: first derive the model into which you can substitute various numerical values.

Solution: (A) The student should make it clear that the answer here is the fraction of the heliocentric sphere at earth's radius that is subtended by the sun-projected area of the earth times the sun's total power output. Given the orbital radius R_S and earth's radius R_E, the desired fraction is –

$$f = \frac{\pi R_E^2}{4\pi R_S^2} = \frac{R_E^2}{4R_S^2} = \frac{3959^2}{4\bullet(91.7e6)^2} = \text{4.66e-10},$$

and the power intercepted by earth is then equal to (4.66e-10)(3.848e26) = 1.79e17W. (50%)

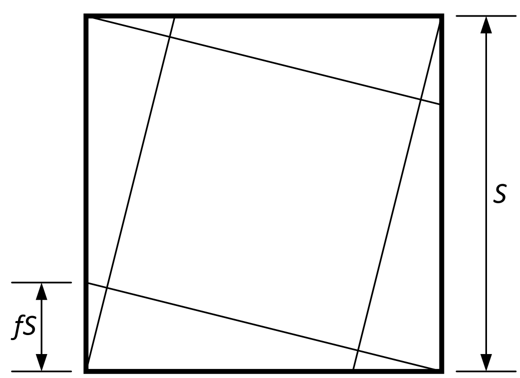
(B) Here the student needs to show that s/he can handle unit conversions, treating the units themselves as algebraic quantities. Given the annual human energy consumption as 110e17Wh, we convert that to earth's power (i.e. rate of energy) consumption by –

$$\frac{\left(\frac{110\text{e}17Wh}{y}\right)}{\left(\frac{365d}{y}\right)\left(\frac{24h}{d}\right)} = \frac{110\text{e}17}{365\bullet 24}W = 1.26\text{e}15W \ ,$$

And then obtain the desired fraction as 1.26e15W/1.79e17W = 0.0015. (50%) That means that the earth intercepts the sun's power at 688 times the rate at which humans consume power (generated mostly by other means than through solar energy).

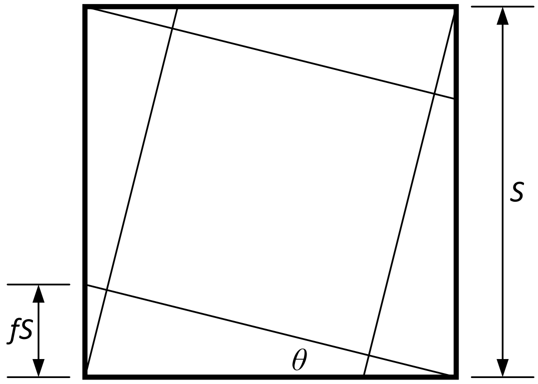
Problem 6: Square in a square (10 points)

A square with side length S is inscribed with lines emanating from each corner to a point located a distance fS on the opposite side as shown, where $0 \leq f \leq 1$. The inscribed lines form the smaller square seen in the figure. (A, 80%) Derive the formula for $R(f)$, the ratio of smaller square's area to that of the larger square, and demonstrate the sanity of your answer. (B, 20%) Calculate R when $S = 5$ and $f = 1/3$?



Solution: The student should immediately realize that, due to the notion of geometrical similarity, the value of S may be set to unity, and that it will not be involved in deriving R. (This subtle hint was given in the problem statement that required the formula $R(f)$ with no S in sight.)

(A) The derivation becomes straightforward with the introduction of $\theta = \tan^{-1} f$ as shown in the nearby figure.



The length of the inscribed lines is then $\sqrt{1+f^2}$. This lets us proceed by inspection to derive the length L of the small square's side. We first note that $\sin\theta = f\cos\theta$.

$$\begin{aligned} L &= \sqrt{1+f^2} - f\sin\theta - f\cos\theta = \sqrt{1+f^2} - f\left(\sin\theta + \cos\theta\right) \\ &= \sqrt{1+f^2} - f\left(f\cos\theta + \cos\theta\right) \\ &= \sqrt{1+f^2} - f\left(1+f\right)\cos\theta = \sqrt{1+f^2} - f\left(1+f\right)\cos\left(\tan^{-1} f\right) \end{aligned},$$

Then R is simply $L^2/1$ or

$$R(f) = \left[\sqrt{1+f^2} - f\left(1+f\right)\cos\left(\tan^{-1} f\right)\right]^2 \qquad \text{(60\%)}$$

The most direct way to check the sanity of the above equation for $R(f)$ is to let f assume its extreme values at which we know the values of R. At $f = 0$ the smaller and larger squares have equal sides and $R(0) = 1$. This is confirmed from the above since the first term equals one and the second term is zero. At the other extreme we examine R(1) at which all four diagonals intersect at a single point in the center of the unit square, thereby yielding the area of the smaller square as zero. We confirm that $R(1) = 0$ by noting that for $f = 1$ we have $\theta = 45^0$. The second term then contains $\cos 45^0$ and $\cos 45^0 = 1/\sqrt{2}$, this calculates to

$$\begin{aligned} R(f) &= \left[\sqrt{1+f^2} - f\left(1+f\right)\cos\left(\tan^{-1} f\right)\right]^2 \\ &= \left[\sqrt{2} - 1(2)\frac{1}{\sqrt{2}}\right]^2 = \left[\sqrt{2} - \sqrt{2}\right]^2 = 0 \end{aligned} \qquad \text{(20\%)}$$

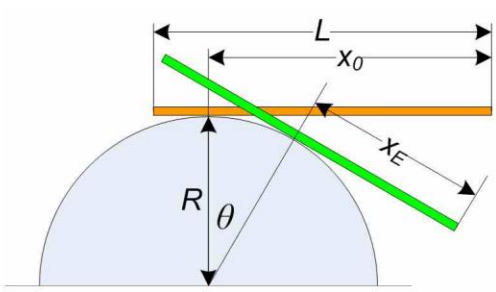
(C) Substituting $f = 1/3$ and calculating yields

$$R(f) = \left[\sqrt{1+\left(\frac{1}{3}\right)^2} - \frac{1}{3}\left(1+\frac{1}{3}\right)\cos\left(\tan^{-1}\frac{1}{3}\right) \right]^2$$

$$= \left[\sqrt{\frac{10}{9}} - \frac{4}{9}\cos(18.435) \right]^2 = [1.111 - 0.4216]^2 = 0.400$$

(20%)

Problem 7: Stick Balanced on Ball (15 points) Few people notice that it is easy to balance a stick on a ball, and fewer inquire why that is so. Just place the stick on the ball so that its approximate center point touches near the top of the ball, and the stick will automatically tilt in such a way that it comes to a point of equilibrium at an angle to the horizontal. Suppose you have a stick of constant linear density ρ (kg/m) that is L meters long. You also have a sphere (ball) of radius R (m) made of a non-deforming substance such that it prevents the stick from sliding along its surface. You place the stick on the top of the ball so that its original tangent point is x_0 from one end, and you slowly release the stick so that it comes to rest (equilibrium) at some angle θ from the vertical. Let $x_0 > L/2$ with no loss of generality. The figure shows the problem set up with the original and equilibrium positions of the stick in the 'plane of action'.



(A, 40%) Derive the formula that expresses the angle θ and the length x_E to the tangent point at equilibrium in terms of the given parameters. (B, 50%) Discuss how θ and x_E vary as x_0 approaches L. Remember that the stick cannot slide on the ball as long as $\theta < 90^{\circ}$. (C, 10%) Compute the value of θ and x_E for $L = 1.5$ m, $R = 1.1$ m, $\rho = 3$ kg/m, $x_0 = 0.9$ m. Hint: Moments and torques are measured the same way.

Solution: (A) The equilibrium point is reached when the moments at the tangent point balance each other. Within certain limits this system is stable in the sense that if the ball is rotated in the plane of action (in which the stick lays) then the negative feedback from the unbalanced moments is such that the stick seeks out a new value of θ and its corresponding x_E where the moments are again opposite and equal. The solution to this problem is simpler than it looks. We start by writing down the relationship between x_0 and x_E where θ is in radians.

$$x_0 = R\theta + x_E > L/2$$

$$x_E = x_0 - R\theta \rightarrow \theta = \frac{x_0 - x_E}{R}$$

Then balancing the gravity induced moments around the equilibrium tangent point, we have

$$\rho g\left(L-x_E\right)\left(\frac{L-x_E}{2}\cos\theta\right)=\rho g x_E\left(\frac{x_E}{2}\cos\theta\right)$$

$$x_E^2=\left(L-x_E\right)^2=L^2-2Lx_E+x_E^2\rightarrow x_E=\frac{L}{2}$$

This demonstrates what should be obvious by inspection, that the equilibrium point is independent of θ and always occurs at the half-way point on the stick. (40%)

(B) Substituting the solution for x_E into the above equation for θ gives

$$\theta=\frac{x_0-L/2}{R}=\frac{2x_0-L}{2R}<\frac{\pi}{2}$$

$$\lim_{x_0\rightarrow L}\theta=\theta_{\max}=\frac{L}{2R}<\frac{\pi}{2}\Rightarrow L<\pi R,$$

which says that for sticks shorter than half the circumference of the ball, the stick will rotate to an equilibrium point whose maximum angle is less that $\pi/2$ or 90 degrees as the original tangent point x_0 approaches one end of the stick. If $L > \pi R$, then we can derive the maximum value of x_0, its related x_E, and θ from the constraint that the equilibrium doesn't exceed $\pi/2$ (and the stick drops off the ball).

$$\frac{2x_{0,\max}-L}{2R}\leq\frac{\pi}{2}\rightarrow x_{0,\max}\leq\frac{L+\pi R}{2}$$

$$\therefore x_{E,\max}=x_{0,\max}-R\frac{\pi}{2}=\frac{L+\pi R}{2}-\frac{\pi R}{2}=\frac{L}{2},\ \text{and confirming}$$

$$\theta_{\max}=\frac{1}{R}\left[\frac{L+\pi R}{2}-\frac{L}{2}\right]=\frac{\pi}{2}.$$

(50%)

(C) From the above development we can immediately say that $x_E = 0.75$ m, and

$$\theta=\frac{x_0-x_E}{R}=\frac{0.9-0.75}{1.1}=0.136\text{ rad.}$$

(10%)

Problem 8: Differentiation (5 points)

Given $y=\dfrac{x^2+\ln x}{\left(\sin 3x+4\right)^2}$, find $\dfrac{dy}{dx}$. (Hint: express y as a function of multiple subfunctions of x and use the appropriate derivative formula for that expression in order to simplify your work.)

Solution: The most obvious re-expression of y is $y = u(x)/v(x)$, giving

$$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx} = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$$

Substituting and taking the elementary derivatives

$$\frac{du}{dx} = \frac{d}{dx}\left(x^2 + \ln x\right) = 2x + \frac{1}{x}, \quad \frac{dv}{dx} = \frac{d}{dx}(\sin 3x + 4)^2 = 2(\sin 3x + 4)(3\cos 3x)$$

$$\text{then } \frac{dy}{dx} = \frac{1}{(\sin 3x + 4)^4}\left\{(\sin 3x + 4)^2\left(2x + \frac{1}{x}\right) - 6\left(x^2 + \ln x\right)(\sin 3x + 4)(\cos 3x)\right\}.$$

Performing the algebra and collecting terms give the final answer.

$$\frac{dy}{dx} = \frac{\left(2x + \frac{1}{x}\right)}{(\sin 3x + 4)^2} - \frac{6\left(x^2 + \ln x\right)(\cos 3x)}{(\sin 3x + 4)^3}.$$

Problem 9: Iterative Update of Cumulative Interest (10 points)

A school bond has been paying varying amounts of annual interest i_k, $k = 1, 2, \ldots$. During the first N years the bond has paid a cumulative interest I_N that is calculated from

$$I_N = \prod_{k=1}^{N}(1 + i_k) - 1 \ .$$

To update the cumulative interest, we desire to avoid having to store all the i_k and recompute the above product every time when the new annual interest rate becomes known. (A, 70%) Derive the formula that lets us compute I_{N+1} as a function of only I_N and i_{N+1}. (B, 30%) Demonstrate that your formula is correct.

Solution: (A) Begin by formally expressing I_{N+1}, then extract I_N from that, and finally collect terms.

$$\begin{aligned} I_{N+1} &= \prod_{k=1}^{N+1}(1 + i_k) - 1 = (1 + i_{N+1})\underbrace{\prod_{k=1}^{N}(1 + i_k)}_{I_N + 1} - 1 \\ &= (1 + i_{N+1})(I_N + 1) - 1 = 1 + I_N + i_{N+1}I_N + i_{N+1} - 1 \\ &= (1 + i_{N+1})I_N + i_{N+1} \end{aligned}$$

(70%)

(B) Let N = 2 and expand I_2, then compare with derived formula in (A).

$$I_2 = \prod_{k=1}^{2}(1+i_k)-1 = (1+i_1)(1+i_2)-1$$
$$= 1+i_2+i_1+i_1 i_2 - 1 = (1+i_2)\,i_1 + i_2$$
$$= (1+i_2)\underbrace{(1+i_1-1)}_{I_1+1} + i_2 \quad \text{QED}$$

(30%)

